

# Lagrangian Floer theory, from geometry to algebra and back again

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**Abstract.** We survey various aspects of Floer theory and its place in modern symplectic geometry, from its introduction to address classical conjectures of Arnold about Hamiltonian diffeomorphisms and Lagrangian submanifolds, to the rich algebraic structures captured by the Fukaya category, and finally to the idea, motivated by mirror symmetry, of a “geometry of Floer theory” centered around family Floer cohomology and local-to-global principles for Fukaya categories.

**1 Arnold’s conjectures and Floer (co)homology.** Since their introduction by Andreas Floer almost forty years ago to study fundamental questions in symplectic geometry and low-dimensional topology, Floer homology theories have been a major driver of progress in those areas of mathematics.

Floer’s work in symplectic geometry was motivated by key conjectures of Arnold about *fixed points of Hamiltonian diffeomorphisms* and *intersections of Lagrangian submanifolds* [16]. We briefly recall some basic notions in symplectic geometry (see [30] for a detailed treatment). First of all, a **symplectic form** is a closed non-degenerate 2-form  $\omega$  on a smooth manifold  $M$ . For instance:

- $\mathbb{R}^{2n} = \mathbb{C}^n$  carries the standard symplectic form  $\omega_0 = \sum dx_i \wedge dy_i$ ;
- any cotangent bundle  $M = T^*N$  has an exact symplectic form  $\omega = d\lambda$ , given in coordinates  $(q_i)$  on  $N$  and dual coordinates  $(p_i)$  on the fibers by  $\omega = \sum dp_i \wedge dq_i$ ; here  $\lambda = \sum p_i dq_i$  is the canonical Liouville form;
- $\mathbb{C}P^n$ , or by restriction, any complex projective variety, with the Fubini-Study Kähler form.

A smooth function  $H : M \rightarrow \mathbb{R}$  on a symplectic manifold  $(M, \omega)$  determines a **Hamiltonian vector field**  $X_H$ , characterized by the property that  $\omega(\cdot, X_H) = dH$ . The flows generated by time-dependent Hamiltonian vector fields are called **Hamiltonian diffeomorphisms**; they form a subgroup  $\text{Ham}(M, \omega)$  of the symplectomorphism group  $\text{Symp}(M, \omega) = \{\varphi \in \text{Diff}(M) \mid \varphi^*\omega = \omega\}$ . For instance, the classical mechanical system consisting of a point mass moving in a potential  $V(q)$  on the manifold  $N$  is described by the dynamics generated by the Hamiltonian  $H(q, p) = \frac{1}{2}|p|^2 + V(q)$  on the phase space  $T^*N$ .

Arnold’s conjecture states that the number of fixed points of a Hamiltonian diffeomorphism of a compact symplectic manifold  $(M, \omega)$  is at least the minimal number of critical points of a smooth function on  $M$ . Moreover:

**CONJECTURE 1.1 (Arnold’s conjecture for non-degenerate Hamiltonians).** *Given a compact symplectic manifold  $(M, \omega)$  and  $\varphi \in \text{Ham}(M, \omega)$  with non-degenerate fixed points,*

$$(1.1) \quad \#\text{Fix}(\varphi) \geq \sum_{i=0}^{\dim M} \dim H^i(M, \mathbb{Q}).$$

Arnold’s conjecture in this formulation was first proved by Floer for monotone symplectic manifolds [39, 40], then in the semi-positive setting by Hofer-Salamon [64] and Ono [80], and in full generality by Fukaya-Ono [46] and Liu-Tian [71] (see also Pardon [82] or Filippenko-Wehrheim [38] for more modern treatments).

The inequality (1.1) is easily seen to hold for Hamiltonian diffeomorphisms generated by a time-independent Morse function, since  $X_H$  vanishes at every critical point of  $H$ . The general case however is considerably more difficult, and the proof crucially relies on the construction of Hamiltonian Floer (co)homology,<sup>1</sup> which we now sketch. (See [18] and [72, Chapter 12] for detailed texts about the construction.)

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<sup>1</sup>As with Morse (co)homology, the only differences between Floer homology and cohomology are in the grading conventions and in deciding which end of a trajectory is the output. We will consistently prefer cohomology, due to its ring structure.

**1.1 Hamiltonian Floer (co)homology and Arnold's conjecture.** Given a (time-dependent, 1-periodic, non-degenerate) Hamiltonian  $H$  and an  $\omega$ -compatible almost-complex structure  $J$  on  $M$  (i.e.,  $J \in \text{End}(TM)$  satisfying  $J^2 = -1$  and such that  $\omega(\cdot, J\cdot)$  is a Riemannian metric), the Floer complex  $CF^*(M, H; J)$  is the free module (over a suitable coefficient ring or field) generated by the set  $\mathcal{X}(H)$  of 1-periodic orbits of  $X_H$ , equipped with a differential which counts solutions to a perturbed Cauchy-Riemann equation known as **Floer's equation**. Namely, we consider maps  $u : \mathbb{R} \times S^1 \rightarrow M$ ,  $(s, t) \mapsto u(s, t)$ , which satisfy

$$(1.2) \quad \frac{\partial u}{\partial s} + J \left( \frac{\partial u}{\partial t} - X_{H_t} \right) = 0,$$

with finite energy  $E(u) = \iint \|\partial u / \partial s\|^2 ds dt$ , and which converge as  $s \rightarrow \pm\infty$  to given periodic orbits  $x_{\pm} \in \mathcal{X}(H)$ . Floer's equation is (formally) the gradient flow of the **action functional** on the free loop space of  $M$ , defined by

$$(1.3) \quad \mathcal{A}_H(x) = \int (H_t(x(t)) - \lambda(\dot{x}(t))) dt$$

when  $\omega = d\lambda$  is exact (and in this case  $E(u) = \mathcal{A}_H(x_-) - \mathcal{A}_H(x_+)$ ); in the non-exact case  $\mathcal{A}_H$  is multivalued, but the intuition remains the same: the 1-periodic orbits of  $X_H$  are the critical points of  $\mathcal{A}_H$ , and the Floer (co)homology of  $H$  is (formally) the Morse(-Novikov) (co)homology of  $\mathcal{A}_H$ .

For generic  $H$  and  $J$ , the moduli space  $\mathcal{M}(x_+, x_-; [u], J)$  of Floer trajectories (i.e., solutions to (1.2)) connecting given generators  $x_{\pm} \in \mathcal{X}(H)$  and in a given homotopy class  $[u]$ , up to  $\mathbb{R}$ -translation in the  $s$  direction, is a finite dimensional manifold (of dimension determined by the Conley-Zehnder index).

As an application of Gromov compactness,  $\mathcal{M}(x_+, x_-; [u], J)$  can be compactified by adding (1) **broken trajectories**, i.e. tuples of Floer trajectories from  $x_+$  to some other generator, and so on, to  $x_-$ , and (2) nodal configurations consisting of Floer trajectories (of lower energy) together with one or more  $J$ -holomorphic **sphere bubbles** attached to each other at nodes.

When all the strata of the compactified moduli space  $\overline{\mathcal{M}}(x_+, x_-; [u], J)$  are regular (i.e., smooth and of the expected dimension), sphere bubbling only occurs in real codimension 2, and the (codimension 1) boundary corresponds exactly to two-step broken trajectories:

$$(1.4) \quad \partial \overline{\mathcal{M}}(x_+, x_-; [u], J) = \bigsqcup_{\substack{y \in \mathcal{X}(H) \\ [u] = [u_-] \# [u_+]}} \overline{\mathcal{M}}(y, x_-; [u_-], J) \times \overline{\mathcal{M}}(x_+, y; [u_+], J).$$

The **Floer differential**  $\partial : CF^*(M, H; J) \rightarrow CF^*(M, H; J)$  is then defined by counting trajectories in zero-dimensional moduli spaces, with signs and with suitable weights  $w([u])$ :

$$(1.5) \quad \partial x_+ = \sum_{x_-, [u]} \#(\mathcal{M}(x_+, x_-; [u], J)) w([u]) x_-.$$

(In the non-exact setting, one typically works over the **Novikov field** of formal power series with real exponents,  $\mathbb{K} = \Lambda_k := \{ \sum a_i T^{\lambda_i}, a_i \in k, \lambda_i \in \mathbb{R}, \lambda_i \rightarrow +\infty \}$ , and solutions are weighted by energy:  $w([u]) = T^{E([u])}$ .) As in classical Morse theory, by considering (1.4) for 1-dimensional moduli spaces one finds that  $\partial^2 = 0$ , which allows one to define the Floer cohomology  $HF^*(M, H; J) = \text{Ker}(\partial) / \text{Im}(\partial)$ .

When  $M$  is compact, the Floer cohomology is independent of the choice of compatible almost-complex structure  $J$  and Hamiltonian  $H$ , and it is isomorphic to the (quantum) cohomology of  $M$  via the PSS (Piunikhin-Salamon-Schwarz) isomorphism. (This is in fact a ring isomorphism, where the product structure on Floer cohomology is defined by counting solutions of a suitable analogue of (1.2) for maps from a pair of pants to  $M$ .) Thus, when Hamiltonian Floer cohomology is well-defined, the Arnold conjecture follows from the fact that the rank of the Floer complex is bounded from below by the rank of its cohomology. A key technical point that restricted the level of generality of the early proofs, however, is achieving regularity for moduli spaces of configurations that include sphere bubbles so that (1.4) holds (see [72] for the argument in the semi-positive case); or, failing that, constructing **virtual fundamental classes** for the moduli spaces [46, 71, 82, 45].

*Remark 1.2.* Outside of the monotone setting, the construction of Floer cohomology usually requires working over a field of characteristic zero (due to the moduli spaces being orbifolds rather than manifolds in the presence of multiply covered bubbles), thus yielding the bound (1.1) by the cohomology of  $M$  with rational coefficients. Two recent advances make it possible to improve on this bound when the cohomology of  $M$  has torsion. First,

Abouzaid and Blumberg have shown that Floer moduli spaces admit fundamental classes not only in ordinary cohomology but also in Morava K-theory [8], which leads to a bound by the rank of the cohomology of  $M$  with coefficients in a field of characteristic  $p$ . Second, Bai-Xu [26] and Rezkikov [88] have constructed a version of Floer theory over the Novikov ring with  $\mathbb{Z}$  coefficients (by dealing with multiply covered configurations in a specific manner), which gives a lower bound by the total Betti numbers over  $\mathbb{Z}$  (including torsions of all characteristics).

Floer cohomology can also be defined on noncompact symplectic manifolds, subject to suitable conditions on the manifold and on the Floer data  $(H, J)$ , so as to control the behavior of solutions to (1.2) near infinity. A common setup is that of **Liouville manifolds**, i.e. exact symplectic manifolds whose ends are modelled on the positive symplectization of a contact manifold  $(N, \alpha)$ , namely  $N \times [1, +\infty)$  with the symplectic form  $\omega = d(r\alpha)$ . (For instance, cotangent bundles are Liouville, and so are affine complex algebraic varieties.) One then considers Hamiltonians which grow linearly with  $r$  at infinity (whose 1-periodic orbits near infinity correspond to closed orbits of the Reeb vector field of the contact form  $\alpha$  on  $N$ ); the Floer cohomology of such  $H$  depends substantially on its slope at infinity, but the direct limit of the Floer cohomology groups for Hamiltonians of increasingly large slope, called **symplectic cohomology** and denoted  $SH^*(M)$ , is independent of auxiliary choices.

**1.2 Lagrangian Floer (co)homology.** Lagrangian Floer theory was developed by Floer [39] to approach another conjecture of Arnold about intersections of Lagrangian submanifolds. Recall that an  $n$ -dimensional submanifold of a  $2n$ -dimensional symplectic manifold  $(M, \omega)$  is said to be **Lagrangian** if  $\omega|_L = 0$ . For example, the zero section is a Lagrangian submanifold of the cotangent bundle  $T^*N$  with its standard symplectic form, and in fact, by the Weinstein neighborhood theorem, this is the universal example: a Lagrangian submanifold  $L \subset M$  always admits a tubular neighborhood which is symplectomorphic to a neighborhood of the zero section in  $T^*L$ .

The **Arnold-Givental conjecture** and other Lagrangian versions of Arnold's conjecture state that, under suitable assumptions, the image  $\varphi(L)$  of a Lagrangian submanifold  $L$  under a Hamiltonian diffeomorphism  $\varphi \in \text{Ham}(M, \omega)$ , when it intersects  $L$  transversely, must do so in a number of points at least equal to the total dimension of the cohomology of  $L$ . The statement doesn't hold unconditionally: for example the unit circle  $S^1 \subset (\mathbb{R}^2, dx \wedge dy)$  is disjoint from its image under the translation generated by a sufficiently large linear Hamiltonian. On the other hand, it holds for the diagonal Lagrangian  $\Delta \subset (M \times M, -\omega \oplus \omega)$ . Since the image of  $\Delta$  under a Hamiltonian diffeomorphism of the form  $\text{id} \times \varphi$  is the graph of  $\varphi$ , the Arnold conjecture for Hamiltonians follows as a special case. Floer proved:

**THEOREM 1.3** (Floer [39]). *Let  $L$  be a compact Lagrangian submanifold of a compact symplectic manifold  $(M, \omega)$ , such that  $\pi_2(M, L) = 0$ . Then for any  $\varphi \in \text{Ham}(M, \omega)$  such that  $\varphi(L)$  meets  $L$  transversely,*

$$\#(L \cap \varphi(L)) \geq \sum_{i=0}^{\dim L} \dim H^i(L, \mathbb{Z}/2).$$

The assumption can be relaxed to only require  $[\omega] \cdot \pi_2(M, L) = 0$ , i.e. the integral of  $\omega$  over any disc bounded by  $L$  is zero; and  $M$  need not be compact (but should be convex at infinity, so Floer theory is well-defined). Thus, Theorem 1.3 applies e.g. to a compact **exact Lagrangian** submanifold of a Liouville manifold  $(M, \omega = d\lambda)$ , i.e. a Lagrangian submanifold  $L$  such that the 1-form  $\lambda|_L$  is exact (e.g., the zero section in a cotangent bundle).

To a pair of Lagrangian submanifolds  $L_0, L_1 \subset (M, \omega)$ , a (time-dependent) Hamiltonian  $H$  such that  $\varphi_H^1(L_0)$  intersects  $L_1$  transversely, and an  $\omega$ -compatible almost complex structure  $J$  on  $M$ , one associates the **Lagrangian Floer complex**  $CF^*(L_0, L_1; H, J)$ , which is a free module (over a suitable coefficient field  $\mathbb{K}$ :  $\mathbb{Z}/2$  in the setting of Theorem 1.3, the Novikov field in non-exact settings, etc.) generated by the set  $\mathcal{X} = \mathcal{X}(L_0, L_1; H)$  of trajectories  $x(t)$  of the Hamiltonian vector field  $X_H$  such that  $x(0) \in L_0$  and  $x(1) \in L_1$ . (Equivalently, since each trajectory is determined by its end point  $x(1)$ , one could also define  $\mathcal{X} = \varphi_H^1(L_0) \cap L_1$ .) This is equipped with the Floer differential, which counts solutions to Floer's equation (1.2), where now the domain of the map  $u$  is the strip  $\mathbb{R} \times [0, 1]$ , and we impose boundary conditions  $u(s, 0) \in L_0$  and  $u(s, 1) \in L_1$  for all  $s \in \mathbb{R}$ . (As in the Hamiltonian case, solutions to (1.2) can also be thought of as gradient flow lines of a suitably defined action functional on the space of paths  $x : [0, 1] \rightarrow M$  which start on  $L_0$  and end on  $L_1$ .)

For generic  $H$  and  $J$ , the moduli space  $\mathcal{M}(x_+, x_-; [u], J)$  of Floer trajectories connecting given generators  $x_{\pm} \in \mathcal{X}$  and in a given homotopy class  $[u]$ , up to  $\mathbb{R}$ -translation, is a finite dimensional manifold (of dimension determined by the Maslov index). As in the Hamiltonian case, the Floer differential  $\partial : CF^*(L_0, L_1; H, J) \rightarrow CF^*(L_0, L_1; H, J)$  is defined by counting trajectories in zero-dimensional moduli spaces (with suitable signs and

weights  $w([u])$  when the coefficient field is not  $\mathbb{Z}/2$  as in the setting of Theorem 1.3):

$$(1.6) \quad \partial x_+ = \sum_{x_-, [u]} \#(\mathcal{M}(x_+, x_-; [u], J)) w([u]) x_-.$$

(When  $\text{char}(\mathbb{K}) \neq 2$ , one should equip  $L_0$  and  $L_1$  with spin structures in order to determine orientations of the moduli spaces so as to be able to count with signs.)

The compactification of  $\mathcal{M}(x_+, x_-; [u], J)$  involves adding broken trajectories (which arise as limits of configurations where energy escapes towards  $s \rightarrow \pm\infty$ ) as well as nodal configurations containing  $J$ -holomorphic sphere or disc bubbles (which arise as limits when energy concentrates near an interior or boundary point of the domain  $\mathbb{R} \times [0, 1]$ ). In general, **disc bubbling** is an **obstruction** to the definition of Lagrangian Floer cohomology, as it is expected to occur in real codimension 1, thereby causing the boundary of the compactified moduli space  $\overline{\mathcal{M}}(x_+, x_-; [u], J)$  to *not* satisfy (1.4). This in turn means that the Floer differential defined by (1.6) does not always square to zero. Thus, excluding the occurrence of disc bubbles (or cancelling them out algebraically) is a prerequisite to the definition of Lagrangian Floer cohomology. (As in the Hamiltonian case, there are also issues of regularity of the moduli spaces, or how to count solutions in the absence of regularity, to be dealt with. In simple cases these can be dealt with by elementary methods, but the general case requires abstract perturbation techniques such as Kuranishi structures or polyfolds; we refer the reader to [45] for one possible treatment.)

In the setting of Theorem 1.3, the assumption that  $[\omega] \cdot \pi_2(M, L_i) = 0$  ensures that  $L_i$  cannot bound any holomorphic discs (since  $J$ -holomorphic curves have positive symplectic area), thereby excluding disc bubbling (as well as sphere bubbling, since it follows that  $[\omega] \cdot \pi_2(M) = 0$ ) and making Floer cohomology well-defined.

For compact Lagrangians, the Floer complexes for different choices of Floer data (i.e., the Hamiltonian  $H$  and the almost-complex structure  $J$ ) are related to each other by *continuation maps* which count solutions to a version of Floer's equation (1.2) where  $J$  and  $H$  depend on  $s$ . In the absence of disc bubbling, the continuation maps are quasi-isomorphisms, so that the Floer cohomology  $HF^*(L_0, L_1)$  is independent of these auxiliary choices. Theorem 1.3 then follows from an explicit comparison between the Floer complex  $CF^*(L, L; H, J)$  for a Hamiltonian  $H$  whose restriction to  $L$  is a small multiple of a given Morse function  $f : L \rightarrow \mathbb{R}$ , and the Morse complex of  $f$ , from which one deduces that  $HF^*(L, L) \simeq H^*(L; \mathbb{K})$  under the assumption that  $L$  is compact and does not bound any  $J$ -holomorphic discs.

In the presence of  $J$ -holomorphic discs, there are situations where disc bubbling can be kept under control and Floer cohomology can still be defined by elementary methods. One of these is the case of monotone Lagrangian submanifolds in monotone symplectic manifolds (i.e., when the Maslov class and the symplectic area of disks are positively proportional to each other), first studied by Oh [79]. However, the definition of Lagrangian Floer cohomology in full generality requires both addressing the regularity issues that may arise with moduli spaces, and coming up with a way to study and, when possible, cancel out the obstructions that come from disc bubbling. This large-scale undertaking is the heart of Fukaya, Oh, Ohta and Ono's monograph [44]. Fukaya-Oh-Ohta-Ono deal with the latter issue by introducing the notion of *bounding cochain*, which we will discuss in the next section. This has the following striking application to the Lagrangian Arnold conjecture:

**THEOREM 1.4** (Fukaya-Oh-Ohta-Ono [44, Theorem H]). *Let  $L$  be a relatively  $\text{spin}^2$  compact Lagrangian submanifold of a compact symplectic manifold  $(M, \omega)$ . Assume that the inclusion map  $i_* : H_*(L, \mathbb{Q}) \rightarrow H_*(M, \mathbb{Q})$  is injective. Then for any  $\varphi \in \text{Ham}(M, \omega)$  such that  $\varphi(L)$  meets  $L$  transversely,*

$$\#(L \cap \varphi(L)) \geq \sum_{i=0}^{\dim L} \dim H_i(L, \mathbb{Q}).$$

The key ingredient that goes into the proof of this result is that the injectivity of  $i_*$  ensures the existence of a bounding cochain  $b \in CF^*(L, L)$  which can be used to algebraically cancel the effects of disc bubbling.<sup>3</sup>

Floer cohomology can also be defined for noncompact Lagrangian submanifolds in noncompact symplectic manifolds, subject to suitable conditions on the geometry at infinity. For example, one can consider exact Lagrangian submanifolds of a Liouville manifold which are cylindrical at infinity, i.e. whose ends are modelled on cylinders  $\Lambda \times [R, +\infty)$  in the symplectization  $(N \times [1, +\infty), \omega = d(r\alpha))$ , where  $\Lambda$  is a *Legendrian* submanifold

<sup>2</sup>A submanifold  $L \subset M$  is said to be relatively spin if  $w_2(TL)$  is in the image of the restriction map  $i^* : H^2(M, \mathbb{Z}/2) \rightarrow H^2(L, \mathbb{Z}/2)$ .

<sup>3</sup>More precisely: there exists a *bulk deformation* of Floer theory for which  $L$  admits a *weak bounding cochain*; see §2.1 and §2.4.

of the contact manifold  $(N, \alpha)$ , i.e. an  $(n - 1)$ -dimensional submanifold such that  $\alpha|_{\Lambda} = 0$ . Using Hamiltonians which grow linearly with  $r$  at infinity, and taking the direct limit of the Floer cohomology groups associated to Hamiltonians of increasingly large slopes, one arrives at the **wrapped Floer cohomology**  $HW^*(L_0, L_1)$  [12]; geometrically, the generators of the wrapped Floer complex consist of intersections of  $L_0$  with  $L_1$  in the interior of the manifold as well as trajectories of  $X_H$  from  $L_0$  to  $L_1$  near infinity, which correspond to *Reeb chords* between the Legendrians  $\Lambda_0$  and  $\Lambda_1$  on which the ends of  $L_0$  and  $L_1$  are modelled. One can also consider other classes of Hamiltonians, giving rise to so-called “partially wrapped” Floer cohomologies; there are natural continuation maps from “less wrapped” to “more wrapped” flavors of Floer cohomology, but they need not be isomorphisms.

We conclude this section with two remarks that highlight the central role of Lagrangian Floer cohomology among Floer theories.

*Remark 1.5.* Hamiltonian Floer cohomology is a special case of Lagrangian Floer cohomology. Given a symplectic manifold  $(M, \omega)$ , the diagonal  $\Delta_M$  is a Lagrangian submanifold in the product  $M \times M$  equipped with the symplectic form  $-\pi_1^*\omega + \pi_2^*\omega$ ; and given  $\varphi \in \text{Ham}(M, \omega)$ , the Hamiltonian diffeomorphism  $\text{id} \times \varphi$  of  $M \times M$  maps  $\Delta_M$  to the graph of  $\varphi$ . There is a natural isomorphism between the Floer complexes  $CF^*(M, H)$  and  $CF^*(\Delta_M, \Delta_M)$  (for suitable choices of Hamiltonian and almost-complex structure on  $M \times M$ ) which intertwines the Floer differentials, so that the Floer cohomologies are isomorphic as well. In fact, Arnold’s bound (1.1) for fixed points of  $\varphi$  is equivalent to the corresponding bound for Lagrangian intersections of  $\Delta_M$  and its image under  $\text{id} \times \varphi$ , and so Conjecture 1.1 also follows from Theorem 1.4.

*Remark 1.6.* In a different direction, low-dimensional topology has greatly benefitted from the introduction and study of various Floer-type invariants of 3-manifolds and knots and links in them. Among these, Ozsváth and Szabó’s **Heegaard-Floer homology** [81] is most directly related to Lagrangian Floer theory: starting from a Heegaard splitting of a 3-manifold  $Y$  into two genus  $g$  handlebodies glued to each other along their boundary surface  $\Sigma$ , the Heegaard-Floer homology of  $Y$  is essentially the Lagrangian Floer cohomology of a pair of Lagrangian tori (associated to the two handlebodies) in the symmetric product  $\text{Sym}^g(\Sigma)$  (or rather, in the complement of the divisor  $\{z\} \times \text{Sym}^{g-1}(\Sigma)$  for the invariant  $\widehat{HF}(Y)$ , or using coefficient weights that keep track of intersection numbers with that divisor for the invariants  $HF^\pm(Y)$ ). Other Floer invariants of 3-manifolds (instanton Floer homology, monopole Floer homology) are gauge-theoretic in nature (i.e., they consider PDEs involving connections on certain bundles over 3-manifolds and their products with  $\mathbb{R}$ ); these are related to Lagrangian Floer theory via the **Atiyah-Floer conjecture** [17], which, given a Heegaard splitting of  $Y$  as above, relates the instanton Floer homology of  $Y$  to the Lagrangian Floer homology of a pair of Lagrangian submanifolds in a moduli space of flat connections over the surface  $\Sigma$ . See e.g. the work of Salamon-Wehrheim [90] and Daemi-Fukaya-Lipyanskiy [36] for progress towards the conjecture. In fact, even invariants that aren’t obviously Floer-theoretic in nature have been shown to admit interpretations in terms of Lagrangian Floer theory: for example, Khovanov homology [13], or potentially a broader range of knot homology theories [15].

## 2 The algebra of Floer theory: the Fukaya category

**2.1 Operations on Floer complexes.** While Floer theory on its own is already immensely useful, its full power comes from the rich algebraic structures it carries. These are defined by counts of solutions to Floer’s equation on more general domains than the strips and cylinders encountered in the previous section.

Given a Riemann surface  $S$  with boundary and punctures, a 1-form on  $S$  with values in Hamiltonian vector fields on  $M$ , i.e.  $X_H \otimes \beta$  where  $\beta \in \Omega^1(S, \mathbb{R})$  and  $H \in C^\infty(S \times M, \mathbb{R})$ , and an almost-complex structure  $J$  on  $M$  (possibly varying over  $S$  as well), one can consider the moduli space of solutions to Floer’s equation

$$(2.1) \quad (du - X_H \otimes \beta)_J^{0,1} = 0,$$

with Lagrangian boundary conditions along  $\partial S$ , of finite energy

$$(2.2) \quad E(u) = \|du - X_H \otimes \beta\|_{L^2}^2 = \int_S u^*\omega - u^*dH \wedge \beta,$$

and asymptotic to given trajectories of  $X_H$  near the punctures of  $S$ . Near the punctures the Hamiltonian perturbation  $X_H \otimes \beta$  is required to be consistent with the choices made in the definition of Lagrangian or Hamiltonian Floer cohomology, i.e. of the form  $X_{H_t} \otimes dt$  in local coordinates  $z = s + it$  identifying each end of  $S$  near a boundary (resp. interior) puncture with a semi-infinite strip (resp. cylinder), so that (2.1) reduces to (1.2).

Letting the choice of domain  $S$  vary over a suitable moduli space of Riemann surfaces with fixed topology, and considering solutions of Floer's equation which converge to given trajectories of  $X_H$  at the punctures of  $S$ , we arrive at moduli spaces of solutions to (2.1), which can be used to define operations on Floer complexes via weighted counts of solutions in moduli spaces of expected dimension zero. (The inputs of the operation correspond to the "positive" strip-like or cylindrical ends, i.e. those in which the local coordinate  $s$  goes to  $+\infty$ , while the outputs correspond to the "negative" ends, those at which  $s \rightarrow -\infty$ .)

These moduli spaces can be compactified by allowing for broken configurations (i.e., letting the domain  $S$  degenerate to a disjoint union of Riemann surfaces, equipped with matching pairs of strip-like or cylindrical ends at which the solutions to Floer's equation on the various components converge to the same trajectories of  $X_H$ ), as well as degenerations of the domain and/or  $J$ -holomorphic disc or sphere bubbling. The codimension 1 boundary strata of the compactified moduli spaces typically consist of two-component configurations; if one chooses the Floer perturbation data in a manner that behaves consistently with respect to these degenerations, the boundary can then be identified with a union of products of simpler moduli spaces. By considering the 0-dimensional boundaries of 1-dimensional moduli spaces, one arrives at algebraic relations satisfied by the operations of the "open-closed Floer TQFT". See e.g. [92, chapter 8] for more details.

In the setting of Hamiltonian Floer cohomology and symplectic cohomology, the most important operation is the **pair-of-pants product**, which is defined by counting solutions to Floer's equation on a 3-punctured Riemann sphere (i.e., a pair of pants), with two of the punctures corresponding to inputs of the product operation and the third one to its output. The chain-level product satisfies the Leibniz rule with respect to the Floer differential, and the induced product on Floer cohomology (or symplectic cohomology) is associative and commutative (in the graded sense). Moreover, the PSS construction (or an analysis of the limit for  $C^2$ -small time-independent Hamiltonians) shows that (over suitable coefficients) the Floer cohomology ring of a compact symplectic manifold is isomorphic to its quantum cohomology.

Likewise, there is a product operation on Lagrangian Floer cohomology, which comes from counting solutions to Floer's equation on a disc with three boundary punctures. Counting (with appropriate weights) such discs whose boundary arcs map to given Lagrangian submanifolds  $L_0, L_1, L_2$  and whose boundary punctures converge to generators of the Floer complexes  $CF^*(L_0, L_1)$ ,  $CF^*(L_1, L_2)$  at the inputs and  $CF^*(L_0, L_2)$  at the output (we omit the Hamiltonian perturbations from the notation), one obtains the chain-level product

$$\mu^2 : CF^*(L_1, L_2) \otimes CF^*(L_0, L_1) \rightarrow CF^*(L_0, L_2),$$

which in the absence of disc bubbling satisfies the Leibniz rule with respect to the differentials and defines a unital associative product on the Floer cohomology groups.

The chain-level product is only associative up to homotopy: the Floer differential  $\partial = \mu^1$  and the product  $\mu^2$  are part of a sequence of  $A_\infty$ -**operations** (higher products) on Lagrangian Floer complexes,

$$(2.3) \quad \mu^d : CF^*(L_{d-1}, L_d) \otimes \cdots \otimes CF^*(L_0, L_1) \rightarrow CF^*(L_0, L_d)$$

for  $d \geq 0$ , counting solutions to Floer's equation on a disc with  $d+1$  boundary punctures ( $d$  inputs and one output), with the successive boundary arcs mapping to  $L_0, L_1, \dots, L_d$ . (When the Floer complexes are  $\mathbb{Z}$ -graded, e.g. when the first Chern class  $c_1(TM)$  vanishes and the Lagrangian submanifolds  $L_0, \dots, L_d$  have vanishing Maslov class, the operation  $\mu^d$  has degree  $2-d$ , reflecting the fact that the moduli space of conformal structures on a  $d+1$ -pointed disc, which compactifies to the Stasheff associahedron, has dimension  $d-2$ .) These operations satisfy the curved (i.e., with  $\mu^0$ )  $A_\infty$ -relations

$$(2.4) \quad \sum_{\substack{0 \leq k \leq d \\ 0 \leq j \leq d-k}} (-1)^* \mu^{d+1-k}(x_d, \dots, x_{j+k+1}, \mu^k(x_{j+k}, \dots, x_{j+1}), x_j, \dots, x_1) = 0,$$

where  $*$  =  $\deg(x_1) + \cdots + \deg(x_j) - j$ . These relations express the fact that the boundaries of 1-dimensional moduli spaces of solutions to Floer's equation on the disc correspond to pairs of discs with the output of one matching an input of the other; these can arise from nodal degenerations of the domain at the boundary of the Stasheff associahedron (the terms in (2.4) with  $2 \leq k \leq d-1$ ), breaking off of a Floer strip at an input ( $k=1$ ) or at the output ( $k=d$ ), or disc bubbling ( $k=0$ ). When  $\mu^0 = 0$ , the first few relations (for  $d=1, 2, 3$ ) state respectively that the differential  $\partial = \mu^1$  squares to zero, the product  $\mu^2$  satisfies the Leibniz rule, and  $\mu^2$  is associative up to an explicit homotopy given by  $\mu^3$ .

The Floer differential generally fails to square to zero for Lagrangian submanifolds with  $\mu^0 \neq 0$ . Bounding cochains were introduced by Fukaya et al. [44] as a way to algebraically deform the  $A_\infty$ -operations to cancel  $\mu^0$ . Working over the Novikov field, an element  $b \in CF^*(L, L)$  (of odd degree, and of positive Novikov valuation, i.e. with coefficients only involving positive powers of the formal variable  $T$ ) is a **bounding cochain** for  $L$  if

$$\mu_b^0 = \sum_{k \geq 0} \mu^k(b^{\otimes k}) = \mu^0 + \mu^1(b) + \mu^2(b, b) + \cdots = 0 \in CF^*(L, L).$$

A Lagrangian submanifold  $L$  is said to be **unobstructed** if it admits a bounding cochain (or one says that the pair  $(L, b)$  is unobstructed). More generally, a Lagrangian is *weakly unobstructed* if it admits a *weak bounding cochain*, i.e.  $b \in CF^*(L, L)$  such that  $\mu_b^0$  is a scalar multiple  $\lambda 1_L$  of the unit in  $CF^*(L, L)$  for some  $\lambda \in \mathbb{K}$ .

Given Lagrangian submanifolds  $L_0, \dots, L_d$  and bounding cochains  $b_i \in CF^*(L_i, L_i)$  for  $i = 0, \dots, d$ , one can modify the Floer operations (2.3) to

$$(2.5) \quad \mu_b^d(x_d, \dots, x_1) = \sum_{k_0, \dots, k_d \geq 0} \mu^{d+k_0+\dots+k_d}(b_d^{\otimes k_d}, x_d, \dots, b_1^{\otimes k_1}, x_1, b_0^{\otimes k_0}).$$

These modified operations satisfy the uncurved (i.e., without  $\mu^0$ )  $A_\infty$ -relations; in particular  $(\mu_b^1)^2 = 0$ , so the Floer cohomology  $HF^*((L_0, b_0), (L_1, b_1))$  of a pair of Lagrangian submanifolds equipped with bounding cochains is well-defined. Likewise for weakly unobstructed Lagrangians that share a common value of the constant  $\lambda \in \mathbb{K}$ .

*Remark 2.1.* There are several different possible ways to define the Floer complex  $CF^*(L, L)$  of a compact Lagrangian submanifold with itself. Here we treat this case identically to the Floer complex of a pair of distinct Lagrangians, by picking a Hamiltonian perturbation (chosen generically so that  $\varphi_H^1(L)$  meets  $L$  transversely); this is similar to e.g. the construction in [92]. However, there exist other models where  $CF^*(L, L)$  is defined to consist of singular chains or differential forms on  $L$ , and the corresponding inputs or outputs of Floer operations are not boundary punctures of the domain  $S$  but rather boundary marked points, with input and output data pulled back and pushed forward through evaluation maps at these marked points; this is the approach taken in e.g. [35], [44], [96], etc. Yet another option is to pick a Morse function  $f : L \rightarrow \mathbb{R}$  and define  $CF^*(L, L)$  to be the free module generated by the critical points of  $f$ ; the Floer operations then count (perturbed) “treed  $J$ -holomorphic discs”, i.e. configurations of  $J$ -holomorphic discs with boundary on  $L$ , together with gradient flow lines of  $f$  connecting their boundaries to each other and to the input and output critical points; see e.g. [27], [32]. Each approach has specific advantages and drawbacks; in the end they yield isomorphic Floer cohomology algebras.

**2.2 The Fukaya category.** There are many versions of the Fukaya category in the literature, depending on what kinds of Lagrangian submanifolds are allowed, whether a  $\mathbb{Z}$ -grading is desired (this typically requires  $2c_1(TM) = 0$ ), the possible presence of additional data such as bounding cochains and local systems, etc. In all cases, the goal is to associate to a symplectic manifold an  $A_\infty$ -category  $\mathcal{F}(M)$  whose objects are Lagrangian submanifolds (together with additional data), with morphisms given by Floer complexes and compositions given by the Floer operations (2.3). See e.g. [22] for an overview of the subject.

The simplest version of the Fukaya category, which can be defined over any coefficient field, considers compact exact Lagrangian submanifolds in a Liouville manifold  $(M, \omega = d\lambda)$ . In this case, exactness precludes bubbling, so that unobstructedness holds tautologically without the need to introduce bounding cochains, and a priori estimates on the energy of Floer solutions (independently of homotopy class) make it unnecessary to work over Novikov coefficients. (However, an object of the Fukaya category should still come equipped with a spin structure, so as to orient moduli spaces of Floer solutions, and grading data if desired.) See [92] for a detailed treatment.

Outside of the exact setting, one should restrict oneself to unobstructed (or weakly unobstructed) Lagrangian submanifolds, possibly after equipping them with (weak) bounding cochains. It is also in general necessary to work over a Novikov field and count solutions of Floer’s equation with weights  $T^{E(u)}$  determined by their energy (2.2). Gromov compactness implies that the weighted counts of Floer solutions are well-defined as elements of the Novikov field, even when they involve contributions from infinitely many homotopy classes of discs. (An exception is the case of *Bohr-Sommerfeld* or *balanced* Lagrangian submanifolds in a monotone symplectic manifold, where the energy of discs is controlled by their index.)

In some settings, e.g. for homological mirror symmetry, one may also consider Lagrangian submanifolds equipped with *local systems*. When working over the Novikov field, these are required to be *unitary*, i.e. their holonomy should be of the form  $a_0 + \sum a_i T^{\lambda_i}$  with  $a_0$  invertible and  $\lambda_i > 0$  for all  $i$ . Solutions to Floer’s equation

are then counted with weights  $w([u]) = T^{E(u)} \text{hol}(\partial u)$ , where  $E(u)$  is the energy and  $\text{hol}(\partial u)$  is the product of the holonomies of the local systems along the boundary of the disc (see e.g. [22, Remark 2.11]).

Perhaps more importantly from a geometric perspective, there are a number of different versions of the Fukaya category for noncompact Lagrangian submanifolds in noncompact symplectic manifolds, depending on the conditions imposed on the geometric behavior of the Lagrangians at infinity and on the class of Hamiltonian perturbations used to define Floer complexes. The **wrapped Fukaya category** of a Liouville manifold involves exact Lagrangian submanifolds which are cylindrical at infinity, with morphisms given by wrapped Floer complexes, i.e. the direct limits of Floer complexes with respect to linear-growth Hamiltonian perturbations of increasingly large slope at infinity, and suitably constructed Floer  $A_\infty$ -operations (see e.g. [1, 12, 56] for different approaches). There are also **partially wrapped** Fukaya categories [98, 56, 57], whose objects are required to avoid certain directions at infinity (the “stops”), and morphisms are direct limits of Floer complexes with respect to Hamiltonian perturbations whose flow stays away from the stops. This includes Fukaya-Seidel categories of symplectic fibrations [92, 93], as well as other variants which lend themselves more easily to calculations for certain classes of examples [21, 61, 6] but can likely be translated into the framework of [56, 57]. One could also consider (partially) wrapped Floer theory on more general classes of (not necessarily exact) noncompact symplectic manifolds. The simplest situation is when the geometry at infinity reduces to the Liouville case (see e.g. [89]), but all that is required is some way to bound the geometric behavior of Floer trajectories via maximum principles and/or energy estimates, and a suitable class of Hamiltonian perturbations; see e.g. [6, Section 3].

The study of Kontsevich’s homological mirror symmetry conjecture [65] has led to a large body of work focused on computations of Fukaya categories (ordinary, wrapped, or partially wrapped depending on the geometric setting), with the goal of comparing these with categories of coherent sheaves (or matrix factorizations) associated to mirror spaces. While it is sometimes possible to list all objects of the Fukaya category and directly compute their Floer cohomologies (see e.g. [85]), a more practical approach is usually to identify a collection of objects which generate the Fukaya category. One says that a collection of objects  $\{G_i\}$  **generates** (resp. **split-generates**) an  $A_\infty$ -category  $\mathcal{C}$  if, in a triangulated enlargement of  $\mathcal{C}$  (for instance twisted complexes or modules over  $\mathcal{C}$  [92]), every object is quasi-isomorphic to an iterated mapping cone built from (arbitrarily many copies of) the objects  $G_i$  (resp. a direct summand in an iterated mapping cone). Then  $\mathcal{C}$  admits a fully faithful embedding into the category of (right) modules over the  $A_\infty$ -algebra  $\mathcal{A} = \bigoplus_{i,j} \text{hom}(G_i, G_j)$ , the *Yoneda embedding* taking each object  $T$  to the  $A_\infty$ -module  $\bigoplus_j \text{hom}(G_j, T)$  (equipped with structure maps given by compositions in the category  $\mathcal{C}$ ). In particular, determining the Floer complexes of the (split-)generators and their  $A_\infty$  structure maps is sufficient to determine the whole Fukaya category. See [22] for an informal treatment and [92] for details.

*Remark 2.2.* Floer-type homology invariants of 3-manifolds (resp. knots and links) often admit further categorifications, which associate an algebra to a surface (resp. a configuration of points in the plane), and a module over this algebra to a 3-manifold with boundary (resp. a tangle). The existence of symplectic interpretations of these invariants (cf. Remark 1.6) suggests that the algebras of interest to low-dimensional topologists should be understood as describing the Fukaya categories of the symplectic manifolds that appear in this context. An example where this works well is bordered Heegaard-Floer homology [70]: Lipshitz-Ozsváth-Thurston’s strands algebra is precisely the endomorphism algebra of a collection of generators of the partially wrapped Fukaya category of the symmetric product of a punctured surface, and the bordered Floer modules associated to 3-manifolds with boundary can be viewed as arising from the Yoneda embedding [21]. Another instance is the arc algebra underlying Khovanov homology [13].

A number of generation criteria have been established to help determine when certain objects generate the Fukaya category, starting with Seidel’s results on Fukaya categories in Lefschetz fibrations [92] and Abouzaid’s generation criterion for wrapped Fukaya categories [1], continuing with results on automatic generation in Calabi-Yau mirror symmetry [59], and generation results for (partially) wrapped Fukaya categories of Weinstein manifolds and sectors [57, 31]. Many of the examples we discuss below are in the context of homological mirror symmetry, but first we start with an example that is of independent interest to symplectic topologists.

**2.3 Cotangent bundles and the nearby Lagrangian conjecture.** About twenty years ago, Nadler and Zaslow [78, 75] constructed a quasi-equivalence between the category of constructible sheaves on a compact manifold  $N$  and a certain “unwrapped” Fukaya category of exact (not necessarily compact) Lagrangians in its cotangent bundle  $T^*N$ , under which compact exact Lagrangian submanifolds (with local systems) in  $T^*N$

correspond to local systems on  $N$ . Fukaya-Seidel-Smith independently arrived at essentially the same conclusion using a different approach [47]. This yields:

**THEOREM 2.3** (Nadler, Fukaya-Seidel-Smith [75, 47]). *Let  $N$  be a compact spin manifold, and  $L \subset T^*N$  be a compact exact Lagrangian submanifold which is spin and whose Maslov class vanishes. Then  $L$  is isomorphic to the zero section in the Fukaya category of  $T^*N$ . In particular, the projection from  $L$  to  $N$  has degree  $\pm 1$  and induces an isomorphism  $H^*(L; \mathbb{K}) \simeq H^*(N; \mathbb{K})$  over any coefficient field.*

Meanwhile, the structure of the wrapped Fukaya category of  $T^*N$  was elucidated by Abouzaid:

**THEOREM 2.4** (Abouzaid [2]). *The wrapped Fukaya category of  $T^*N$  is generated by a cotangent fiber  $F = T_q^*N$  (for any  $q \in N$ ). In particular,  $\mathcal{W}(T^*N)$  quasi-embeds into the category of modules over  $\text{End}(F) \simeq C_{-*}(\Omega_q N)$ , the algebra of chains over the based loop space of  $N$ .*

This is worth comparing with earlier work of Viterbo, Abbondandolo-Schwarz and Salamon-Weber showing that the symplectic cohomology  $SH^*(T^*N)$  is isomorphic to the homology of the free loop space  $\mathcal{L}N$ ; see e.g. [104]. (For simplicity we have stated all the above results under the assumption that  $N$  is spin; they continue to hold when  $N$  is not spin if one considers homology with suitably twisted coefficients.)

Returning to compact exact Lagrangians, we mention the following improvement on Theorem 2.3:

**THEOREM 2.5** (Abouzaid-Kragh [11]). *Given any compact smooth manifold  $N$ , and any compact exact Lagrangian submanifold  $L \subset T^*N$ , the projection from  $L$  to  $N$  is a simple homotopy equivalence.*

This is a significant partial result on Arnold’s **nearby Lagrangian conjecture**, which asks whether every compact exact Lagrangian submanifold in  $T^*N$  is Hamiltonian isotopic to the zero section. (By the Weinstein neighborhood theorem, a tubular neighborhood of a Lagrangian submanifold  $N \subset M$  is symplectomorphic to a neighborhood of the zero section in  $T^*N$ , so Arnold’s conjecture indeed constrains nearby Lagrangians.) Arnold’s question remains open in general (though it has been answered positively in a few cases), essentially because, even though Hamiltonian isotopic exact Lagrangian submanifolds are Fukaya isomorphic, it is not clear that the converse should hold. (Outside of the exact setting, there are infinite families of monotone Lagrangian tori in  $\mathbb{C}P^2$  and many other symplectic manifolds which are not Hamiltonian isotopic to each other yet define isomorphic objects of the Fukaya category when equipped with suitable local systems; see e.g. [102].) In particular, Fukaya categories provide a wealth of information about Lagrangian submanifolds in a given symplectic manifold, but they do not directly address the problem of their classification up to Hamiltonian isotopy. (That said, perhaps an even more basic question is whether objects of the Fukaya category even behave in the geometric manner suggested by the algebra; this is harder than one might think, already for surfaces [25].)

That said, there is evidence that Floer theory can provide more refined information, at least if one keeps track of more detailed information about moduli spaces beyond mere counts of isolated solutions to Floer’s equation – starting with the following result of Abouzaid:

**THEOREM 2.6** (Abouzaid [3]). *Every homotopy sphere which embeds as a Lagrangian in  $T^*S^{4k+1}$  bounds a compact parallelizable manifold.*

More recently, Floer homotopy theory has led to the introduction of “spectral Fukaya categories”, which are only beginning to be systematically explored but already have applications to bordism and stable homotopy types of quasi-isomorphic Lagrangians and to smooth structures on nearby Lagrangians; see e.g. [67, 86, 87].

**2.4 Open-closed maps, deformations, and homological mirror symmetry.** Given Lagrangian submanifolds  $L_0, \dots, L_d$  ( $d \geq 0$ ) of  $(M, \omega)$ , moduli spaces of solutions to Floer’s equation on discs with  $d + 1$  boundary punctures (as in Lagrangian Floer theory) and one single interior puncture (as in Hamiltonian Floer theory) can be used to define **open-closed** and **closed-open** maps

$$(2.6) \quad OC^d : CF(L_d, L_0) \otimes \cdots \otimes CF^*(L_0, L_1) \rightarrow CF^*(M, H),$$

$$(2.7) \quad CO^d : CF^*(M, H) \otimes CF^*(L_{d-1}, L_d) \otimes \cdots \otimes CF^*(L_0, L_1) \rightarrow CF^*(L_0, L_d),$$

depending on whether the output of the operation corresponds to the interior puncture or a boundary puncture.

These maps intertwine the Floer differential on  $CF^*(M, H)$  with the differentials on the Hochschild complexes

$$CC_*(\mathcal{F}(M), \mathcal{F}(M)) = \bigoplus_{d \geq 0} \bigoplus_{L_0, \dots, L_d} CF^*(L_d, L_0) \otimes \cdots \otimes CF^*(L_0, L_1) \quad \text{and}$$

$$CC^*(\mathcal{F}(M), \mathcal{F}(M)) = \prod_{d \geq 0} \prod_{L_0, \dots, L_d} \text{hom}(CF^*(L_{d-1}, L_d) \otimes \cdots \otimes CF^*(L_0, L_1), CF^*(L_0, L_d)),$$

thus inducing cohomology-level maps

$$(2.8) \quad OC : HH_{*-\dim_{\mathbb{C}} M}(\mathcal{F}(M), \mathcal{F}(M)) \rightarrow HF^*(M, H) \quad \text{and} \quad CO : HF^*(M, H) \rightarrow HH^*(\mathcal{F}(M), \mathcal{F}(M)).$$

In the compact setting, one can replace  $HF^*(M, H)$  with the quantum cohomology  $QH^*(M)$ ; and in the noncompact setting there are similar maps in wrapped Floer theory, between the symplectic cohomology  $SH^*(M)$  and the Hochschild homology and cohomology of the wrapped Fukaya category  $\mathcal{W}(M)$ . (However, open-closed maps work somewhat differently for partially wrapped Fukaya categories, including Fukaya-Seidel categories.)

Open-closed maps play a key role in the structure of Fukaya categories. For example:

**THEOREM 2.7** (Abouzaid [1], Ganatra [52, 53]). *Let  $(M, \omega)$  be a Liouville manifold, and assume there exists a Hochschild cycle which maps to the unit  $1 \in SH^*(M)$  under the open-closed map. Then:*

1.  $\mathcal{W} = \mathcal{W}(M)$  is split-generated by the objects whose morphisms appear in the given Hochschild cycle;
2.  $CO : SH^*(M) \rightarrow HH^*(\mathcal{W}, \mathcal{W})$  is a ring isomorphism, and  $OC : HH_{*-\dim_{\mathbb{C}} M}(\mathcal{W}, \mathcal{W}) \rightarrow SH^*(M)$  is an isomorphism of modules over  $HH^*(\mathcal{W}, \mathcal{W}) \simeq SH^*(M)$ ;
3.  $\mathcal{W}$  is a homologically smooth (i.e., its diagonal bimodule is perfect) non-compact Calabi-Yau category (i.e., the inverse dualizing bimodule is isomorphic to the diagonal bimodule up to a grading shift);
4. the  $S^1$ -equivariant enhancement of  $OC$  known as the cyclic open-closed map [53] defines isomorphisms from the (positive, negative, periodic) cyclic homology of  $\mathcal{W}$  to (the corresponding variants of) the  $S^1$ -equivariant symplectic cohomology of  $M$ .

*Remark 2.8.* There is more to this story: the symplectic cohomology  $SH^*(M)$  carries the further structure of a homotopy BV-algebra, and the closed-open map is compatible with these additional operations [10, 28].

Similar results hold for Fukaya categories of compact symplectic manifolds. These ingredients (and in particular the cyclic open-closed map) play a key role in Ganatra-Perutz-Sheridan's program to recover enumerative mirror symmetry from homological mirror symmetry for Calabi-Yau varieties [59]. Namely, let  $M, M^\vee$  be a pair of smooth projective Calabi-Yau manifolds which satisfy homological mirror symmetry in the sense that the Fukaya category  $\mathcal{F}(M)$  is quasi-equivalent to the category of coherent sheaves of (a maximally unipotent degeneration of)  $M^\vee$ . Passing to Hochschild cohomology, one obtains a ring isomorphism between  $HH^*(\mathcal{F}(M)) \simeq QH^*(M)$  and  $HH^*(\text{Coh}(M^\vee)) \simeq H^*(M^\vee, \bigwedge^* TM^\vee)$ , which is the starting point of "classical" mirror symmetry. Upgrading this (as first proposed by Barannikov and Kontsevich) to an isomorphism of variations of Hodge structures over formal punctured discs yields a Hodge-theoretic version of mirror symmetry, which in turn has enumerative consequences, such as e.g. the classical mirror symmetry statement about counts of rational curves in the quintic 3-fold [59].

Since Hochschild cohomology governs first-order deformations of the Fukaya category, the closed-open map being an isomorphism implies that every first-order deformation of the Fukaya category has a geometric origin. In fact, a similar geometric construction also yields actual (formal) deformations of the Fukaya category over the Novikov field. Given a class  $\mathfrak{b} \in H^*(M, \mathbb{K})$  (or more generally in Hamiltonian Floer cohomology) (of even degree, and of positive Novikov valuation), the **bulk-deformed**  $A_\infty$  operations  $\mu_{\mathfrak{b}}^d = \mu^d + CO^d(\mathfrak{b}) + \dots$  are defined by counting solutions to Floer's equation on a disc with  $d+1$  boundary punctures (or more if there are also bounding cochains) and any number  $\ell \geq 0$  of (input) interior punctures, with (a chain-level representative of)  $\mathfrak{b}$  inserted at each of those. One then arrives at the bulk-deformed Fukaya category  $\mathcal{F}(M, \omega, \mathfrak{b})$ . (Note that, since the curvature  $\mu_{\mathfrak{b}}^0$  depends on  $\mathfrak{b}$ , so does the notion of bounding cochain; for instance, for the purpose of proving Theorem 1.4 it is enough to find a weak bounding cochain for  $L$  with respect to *some* bulk deformation.)

Another (related) situation where geometric deformations of the Fukaya category naturally occur is when one compares the Fukaya category of a smooth complex projective variety  $M$  to that of the complement  $M^0 = M \setminus D$  of a smooth (or more generally, normal crossings) divisor  $D \subset M$ . Namely, for Lagrangian submanifolds of  $M$

which are disjoint from  $D$ , and using Floer data that preserve positivity of intersections with  $D$ , one can count solutions to Floer’s trajectories in  $M$ , weighing those which intersect the divisor  $n$  times with an additional factor of  $q^n$  (where  $q$  is a formal variable). This gives rise to the **relative Fukaya category**  $\mathcal{F}(M, D)$ , which presents the Fukaya category of  $M$  as a deformation of  $\mathcal{F}(M^0)$  [91, 84]. The first order term of this deformation, which corresponds to discs that intersect  $D$  transversely once, can be expressed as the image under the closed-open map of a certain element in  $SH^*(M^0)$ , sometimes called the Borman-Sheridan class [29].

These considerations have led to an extremely effective approach to homological mirror symmetry for vast classes of examples, starting with Seidel’s work on genus 2 surfaces [94] which start from a description as a compactification of a  $\mathbb{Z}/5$ -cover of the pair of pants, continuing with Sheridan’s landmark result on Calabi-Yau hypersurfaces in projective space [95], and more recently a proof of homological mirror symmetry for Batyrev mirror pairs [55]. (That said, there are many other approaches to calculations of Fukaya categories and proofs of homological mirror symmetry; see e.g. [69], [6] and [101] for some recent examples.)

*Remark 2.9.* To explain the central role of relative Fukaya categories in many proofs of homological mirror symmetry, recall that mirror symmetry relates symplectic enumerative geometry near the *large volume limit* (i.e., viewing formal variables such as the Novikov parameter  $T$  or the relative Fukaya category parameter  $q$  as living in a neighborhood of the origin) to algebraic geometry near the *large complex structure limit* (i.e., for varieties defined over formal power series, corresponding to maximally degenerating families of complex varieties). The complement of a sufficiently ample hypersurface (e.g. a hyperplane section) in a projective variety is a Liouville (in fact, Stein) manifold, whose Fukaya category can be determined by a wide range of techniques, including microlocal sheaf theory or sectorial decompositions (see the next section); the pushout diagrams inherent to sectorial descent [57] for these large volume limits typically correspond under mirror symmetry to gluing closed subschemes to produce a (singular, reducible) large complex structure limit mirror (see e.g. [50, 51]). One can then hope to recover mirror symmetry for the projective variety itself by a deformation process as in the above examples [94, 95, 55], though the process for doing this in general has not been fully elucidated.

*Remark 2.10.* Deformation theory arguments also explain the appearance of **Landau-Ginzburg models**, i.e. spaces equipped with a function called *superpotential*, and their categories of *matrix factorizations*, in mirror symmetry outside of the Calabi-Yau setting. Namely, given a normal crossings anticanonical divisor  $D$  in  $M$  (i.e., Poincaré dual to  $c_1(TM)$ , which we assume to be nef), the complement  $M^0 = M \setminus D$  is Calabi-Yau, and its (wrapped) Fukaya category can often be related to coherent sheaves on a mirror space  $M^\vee$ . Adding back in the divisor  $D$  then deforms the Fukaya category as explained above; however this deformation does not preserve the  $\mathbb{Z}$ -grading on  $\mathcal{F}(M^0)$ , and the first-order deformation class lives in  $HH^0(\mathcal{F}(M^0))$  rather than in degree 2 as in the Calabi-Yau case. Under mirror symmetry, this becomes an element in  $HH^0(\text{Coh}(M^\vee)) \simeq H^0(M^\vee, \mathcal{O}_{M^\vee})$ , i.e. a function on  $M^\vee$ : the superpotential. Concretely, compact unobstructed Lagrangians in  $M^0$  deform to *weakly* unobstructed Lagrangians in  $M$ , with  $\mu^0$  given by a weighted count of holomorphic discs which intersect  $D$  once: the (constant) value of the superpotential on the support of the mirror object. See also [19].

*Remark 2.11.* Another important structural aspect of Fukaya-Floer theory that we haven’t touched on is its functoriality under Lagrangian correspondences: namely, a Lagrangian submanifold in  $(M \times N, -\pi_M^* \omega_M + \pi_N^* \omega_N)$ , or rather an object of its Fukaya category, determines an  $A_\infty$ -bimodule over the Fukaya categories  $\mathcal{F}(M)$  and  $\mathcal{F}(N)$ , often representable by an  $A_\infty$ -functor from  $\mathcal{F}(M)$  to (an enlargement of)  $\mathcal{F}(N)$ . See e.g. [73, 43, 9].

There are also specific functors between the Fukaya categories of a symplectic fibration (resp. sector), its total space (resp. completion), and its fiber. Under mirror symmetry these often correspond to inclusion and restriction of sheaves between a variety, a divisor in it, and the divisor complement; see e.g. [20, §5] and [99, 77, 23, 49, 62].

**3 Towards the geometry of Floer theory.** At this point, we have turned a lot of symplectic geometry into the language of  $A_\infty$  categories and homological algebra. This fits the philosophy of “noncommutative algebraic geometry”, i.e. the idea that any category that shares structural features with derived categories of algebraic varieties should be regarded as a noncommutative algebraic space. And yet, mirror symmetry tells us that many Fukaya categories actually correspond to honest (commutative) algebraic spaces. This means that there should exist a (classical, “commutative”) geometric perspective on Fukaya categories, lending itself to geometric decomposition results and local-to-global principles – the difficulty being that, a priori, Floer theory is non-local, in the sense that even for Lagrangian submanifolds contained in a subset  $U \subset M$  it may involve trajectories which do not remain within  $U$ . In this section we discuss various perspectives that fit into this general philosophy.

**3.1 Floer theory for families of Lagrangians.** Consider a family of Lagrangian submanifolds  $(F_b)_{b \in B}$  in a symplectic manifold  $(M, \omega)$ , parametrized by a space  $B$ : for instance a Lagrangian fibration  $\pi : M \rightarrow B$  (possibly with singular fibers), but one may also consider families in which  $F_b$  has different topology over different strata of  $B$ , etc. Assuming the Lagrangians  $F_b$  define objects of the Fukaya category of  $M$ , one can associate to another object  $L$  the collection of Floer complexes  $CF^*(F_b, L)$ ,  $b \in B$ , and study how these depend on  $b$ .

For instance, Nadler and Zaslow’s work on Fukaya categories of cotangent bundles [75, 78] fits into this philosophy. Consider the cotangent bundle  $M = T^*N$  of a compact manifold  $N$ , and the family of cotangent fibers  $F_q = T_q^*N$ , parametrized by  $N$  itself. Since nearby cotangent fibers are Hamiltonian isotopic to each other, given an object  $L$  of the Fukaya category supported on a compact exact Lagrangian submanifold, a path from  $q_0$  to  $q_1$  in  $N$  determines an isomorphism between  $HF^*(F_{q_0}, L)$  and  $HF^*(F_{q_1}, L)$ , so that the family of Floer cohomologies  $(HF^*(F_q, L))_{q \in N}$  defines a locally constant sheaf (a local system) over  $N$ . By the same process, a non-compact exact Lagrangian submanifold of  $T^*N$ , allowed to approach infinity along the conormal directions to some given stratification of  $N$ , determines a constructible sheaf on  $N$ .

More generally, let  $M$  be a **Weinstein manifold** (i.e., a Liouville manifold whose Liouville vector field is gradient-like; this includes Stein manifolds, e.g. affine complex algebraic varieties). Then  $M$  retracts onto a half-dimensional *skeleton* (or *core*)  $B$ , whose strata are isotropic submanifolds in  $M$ . When  $B$  is sufficiently nice, one can view  $M$  as obtained by attaching the cotangent bundles of the critical (top-dimensional, i.e. Lagrangian) strata of  $B$  onto the subcritical strata (which do not contribute to the Fukaya category) and onto each other. Taking Floer cohomology with cocores (the analogues of cotangent fibers in this setting) turns Lagrangian submanifolds of  $M$  into sheaves on  $B$  (to be suitably interpreted along the singular locus of the skeleton) – with the benefit that the new data is local over  $B$ , in a way that wasn’t a priori obvious for the Fukaya category (and especially for its wrapped variant). This ties in with a conjecture of Kontsevich [66], who proposed that the wrapped Fukaya category of the Weinstein manifold  $M$  should be given by global sections of a certain cosheaf of categories over its skeleton. Nadler further proposed that the wrapped Fukaya category should be modelled by **wrapped microlocal sheaves** over the skeleton [76]. Due to the entirely combinatorial / topological nature of microlocal sheaf theory, the latter categories are (for trained experts at least) eminently more computable than (wrapped or partially wrapped) Fukaya categories – even as the geometric aspects of wrapped Floer theory, and in particular the Hamiltonian dynamics of wrapping, recede out of sight, since wrapped microlocal sheaves are defined in a purely categorical manner as compact objects in the category of microlocal sheaves [76].

Nadler’s proposal has been proven by Ganatra, Pardon and Shende, under the topological assumption that  $M$  admits a *stable polarization* (i.e., the direct sum of  $TM$  with a trivial vector bundle is the complexification of a real vector bundle): namely, the wrapped Fukaya category of a stably polarized Weinstein sector is equivalent to the category of wrapped microlocal sheaves over its skeleton [58]. This validates a posteriori the approach to homological mirror symmetry via calculations of categories of constructible sheaves pursued by various authors during the 2010s (see e.g. [37, 77]). This is also the approach used by Gammage and Shende to verify homological mirror symmetry for hypersurfaces in  $(\mathbb{C}^*)^n$ , in the direction that relates the wrapped Fukaya category of the hypersurface to coherent sheaves on a mirror space [50]. (The other direction, relating coherent sheaves on the hypersurface to the Fukaya category of a mirror, is proved in [6] using very different methods.)

While the above discussion stems from considering families of contractible Lagrangian submanifolds such as cotangent fibers, another setting in which Floer theory for families has led to powerful advances is that of **Lagrangian torus fibrations**, which are at the heart of the Strominger-Yau-Zaslow (SYZ) approach to mirror symmetry [97] (see also [19]). (Lagrangian torus fibrations arise in a variety of contexts, ranging from completely integrable systems and other situations involving symplectic reduction, to degenerations of algebraic varieties to unions of toric varieties.) Given a Lagrangian torus fibration  $\pi : M \rightarrow B$  (possibly with singular fibers), rather than just the fibers  $F_b = \pi^{-1}(b)$ , one considers families of objects of  $\mathcal{F}(M)$  given by the Lagrangians  $F_b$  together with choices of local systems and/or bounding cochains. (One typically first considers the objects supported on the smooth fibers of  $\pi$ , before possibly completing the family with objects supported on or near the singular fibers.) When the fibers  $F_b$  are tautologically unobstructed (do not bound any holomorphic discs in  $M$ ), the appropriate parameter space is the *uncorrected SYZ mirror*  $Y^0 = \{(F_b, \xi)\}$ , the space of pairs consisting of a smooth fiber  $F_b = \pi^{-1}(b)$  ( $b \in B^0 = B \setminus \text{critval}(\pi)$ ), together with a unitary rank 1 local system  $\xi$  over  $F_b$ , which can be characterized by its holonomy  $\text{hol}_\xi \in \text{hom}(\pi_1(F_b), U(1)_\mathbb{K})$ . This carries a “dual” torus fibration  $\pi^\vee : Y^0 \rightarrow B^0$  mapping  $(F_b, \xi)$  to  $b$ , and has a natural structure of analytic space over  $\mathbb{K}$ , for which the Floer weights  $z_\beta = T^{\int_\beta \omega} \text{hol}_\xi(\partial\beta)$  of homotopy classes of discs  $\beta \in \pi_2(M, F_b)$  are locally analytic functions [19, 4, 41].

In the presence of singular fibers and/or holomorphic discs, one should “correct”  $Y^0$  to a moduli space  $Y$  of weakly unobstructed objects of  $\mathcal{F}(M)$  supported on the fibers of  $\pi$  (the *corrected* SYZ mirror). The fibers of  $\pi$  which bound Maslov index 0 holomorphic discs typically concentrate along a union of codimension 1 *walls* in  $B$ , across which the statement that Hamiltonian isotopic Lagrangians define isomorphic objects of the Fukaya category needs to be corrected by a modification of the bounding cochain or of the local system [44]. Regluing the pieces of  $Y^0$  over the chambers of  $B$  via these wall-crossing transformations yields the corrected SYZ mirror  $Y$ . Outside of the Calabi-Yau setting, the fibers of  $\pi$  typically also bound Maslov index 2 discs; these do not require a modification of the mirror geometry, but they make the objects  $(F_b, \xi)$  *weakly unobstructed*; the quantity  $W(F_b, \xi)$  such that  $\mu^0 = W(F_b, \xi) 1_{(F_b, \xi)}$  defines a global analytic function on  $Y$ , the *mirror superpotential*, thus making the SYZ mirror a Landau-Ginzburg model. See e.g. [19, 7, 100, 105].

The **family Floer program** initiated by Fukaya [41] and further developed by Abouzaid [4, 5], Tu [100] and Yuan [105] leverages Fukaya’s observation that the Floer complexes  $CF^*((F_b, \xi), L)$  and their differentials have a locally analytic dependence on  $(F_b, \xi) \in Y$  [41, 42] to construct the *family Floer functor*, from the Fukaya category of  $M$  to complexes of (coherent) analytic sheaves (or matrix factorizations when the objects are only weakly unobstructed) on  $Y$ , under the assumption that the fibration  $\pi$  admits a Lagrangian section. (Otherwise one obtains *twisted* sheaves with respect to some gerbe on  $Y$  [4].) Roughly speaking, the functor associates to an object  $L$  of  $\mathcal{F}(M)$  the family of Floer complexes  $CF^*((F_b, \xi), L)$ ,  $(F_b, \xi) \in Y$ , and to a morphism  $x \in CF^*(L_0, L_1)$  the family of maps from  $CF^*((F_b, \xi), L_0)$  to  $CF^*((F_b, \xi), L_1)$  given by Floer product with  $x$ . (In the weakly unobstructed case, when  $\mu^0_{(F_b, \xi)} = W(F_b, \xi) 1$  and  $\mu^0_L = \lambda 1$ , the differential on the Floer complexes squares to  $(W - \lambda) \text{id}$ , hence the appearance of matrix factorizations.) The family Floer functor was shown by Abouzaid to be a quasi-equivalence in the case where  $\pi : M \rightarrow B$  doesn’t have any singular fibers, thereby proving homological mirror symmetry for such SYZ mirror pairs [5]. To go beyond this, one needs to have a good understanding of homological mirror symmetry for neighborhoods of singular Lagrangians; this has now been done explicitly for the local models that are relevant to SYZ mirror symmetry [14, 48]. One should also note the “localized mirror functor” approach of Cho-Hong-Lau [33, 34], which can in principle be applied to very general singular Lagrangians and their nearby smoothings, at the expense of involving noncommutative language.

Another perspective on SYZ mirror symmetry (and conjecturally on family Floer theory) is that proposed in [24] to deal with the noncommutative corrections to the mirror geometry that can arise for pairs  $(M, D)$  where the anticanonical divisor  $D$  contains rational curves of negative Chern number. The Floer theory of a smooth fiber  $F_b$  of the Lagrangian fibration  $\pi : M \rightarrow B$ , with coefficients in a completion of the group ring  $\mathbb{K}[H_1(F_b)]$  (counting discs with universal holonomy weights), encodes the family of Floer cohomologies for all the objects of  $\mathcal{F}(M)$  supported on  $F_b$  – an idea already used to great effect by Abouzaid in [5]. The completions of  $\mathbb{K}[\pi_1(H_b)]$  assemble into a sheaf  $\mathcal{O}_{an}$  over the smooth locus  $B^0$ , namely the pushforward of the structure sheaf of the uncorrected SYZ mirror  $Y^0$  under the projection  $\pi^\vee : Y^0 \rightarrow B^0$ . Moduli spaces of pseudo-holomorphic discs in  $M$  with boundary in the fibers  $F_b$  (where  $b$  is now allowed to vary over  $B^0$ ) then determine  $A_\infty$ -operations  $\{\mu^k\}_{k \geq 0}$  not just on Floer cochains of a fixed fiber  $F_b$  with coefficients in  $\mathbb{K}[H_1(F_b)]$ , but also on cochains on  $M^0 = \pi^{-1}(B^0) \subset M$  with coefficients in the pullback of  $\mathcal{O}_{an}$ , or equivalently, on  $\mathfrak{C} = C^*(B^0; C^*(F_b) \otimes \mathcal{O}_{an})$ . The curvature  $\mu^0$  of this  $A_\infty$ -algebra encodes in principle all the information needed to understand corrections to the mirror geometry [24].

A conjectural systematic approach to these is as follows.  $H^*(F_b) \otimes \mathbb{K}[H_1(F_b)]$  is isomorphic to the homology of the free loop space  $\mathcal{L}F_b$ , and carries a natural Lie bracket of degree  $-1$  (the Chas-Sullivan string bracket),

$$(3.1) \quad \{z^\gamma \alpha, z^{\gamma'} \alpha'\} = z^{\gamma+\gamma'} (\alpha \wedge (\iota_\gamma \alpha') + (-1)^{|\alpha|} (\iota_{\gamma'} \alpha) \wedge \alpha').$$

Since  $H^*(F_b, \mathbb{R}) \simeq \Lambda^* H^1(F_b, \mathbb{R}) \simeq \Lambda^* TB$ , there is a natural map from  $H^*(F_b) \otimes \mathbb{K}[H_1(F_b)]$  to polyvector fields over (a neighborhood of)  $(\pi^\vee)^{-1}(b) \subset Y^0$ , under which  $\{\cdot, \cdot\}$  corresponds to the Schouten-Nijenhuis bracket. It is conjectured in [24] that, for a suitable model of Floer theory, the curvature  $\mu^0$  of  $\mathfrak{C}$  satisfies the Maurer-Cartan equation  $\delta \mu^0 + \frac{1}{2} \{\mu^0, \mu^0\} = 0$  (or its  $L_\infty$  analogue) with respect to the classical differential and the bracket (3.1). The corrected differential  $\delta + \{\mu^0, \cdot\}$  on  $\mathfrak{C}$  then squares to zero (unlike  $\mu^1$ ). Conjecturally, under mirror symmetry this amounts to deforming the Čech complex of polyvector fields  $C^*(Y^0; \Lambda^* TY^0)$  to arrive at polyvector fields (or their appropriate noncommutative analogue) on the corrected mirror. Moreover, this differential is expected to be part of a homotopy BV-algebra structure on  $\mathfrak{C}$ , which should match that on polyvector fields (or Hochschild cochains) under mirror symmetry. This story should also be compatible with the construction of family Floer functors; a first step can be found in Hoek’s thesis [63], which constructs a functor from Lagrangian sections of  $\pi$  to modules over the curved  $A_\infty$ -algebra  $(\mathfrak{C}, \{\mu^k\})$ .

**3.2 Geometric decompositions and local-to-global principles.** The two constructions discussed in the previous section turn objects of the Fukaya category of a symplectic manifold  $M$  into sheaves on another space (the skeleton  $B$  in the Weinstein setting, the SYZ mirror space  $Y$  in the setting of Lagrangian torus fibrations), which have locality properties. Despite the non-local nature of Lagrangian Floer theory, in both cases one rightfully expects geometric decompositions of  $M$  into suitable pieces to correspond to geometric decompositions of  $B$  and  $Y$ , thus suggesting that the Fukaya category of  $M$  can be computed from those of the pieces. There are two very different manners in which this principle works in practice.

**Covariant functoriality: Liouville sectors.** Liouville sectors, introduced and studied by Ganatra, Pardon and Shende [56, 57], are a class of exact symplectic manifolds with boundary, whose non-compact ends are convex, i.e. modelled on the positive symplectization of a contact manifold with boundary, and which satisfy a set of geometric conditions along their boundary ensuring that (for suitable almost-complex structures) families of  $J$ -holomorphic curves cannot escape through the sectorial boundary. A simple example of sector is the cotangent bundle  $T^*N$  of a manifold with boundary  $N$ ; in this case the sectorial boundary is  $\partial(T^*N) \simeq \mathbb{R} \times T^*(\partial N)$ . More generally, *Weinstein sectors* are built by attaching cotangent bundles of critical strata of their skeleton (along part of their boundary only) onto a subcritical sector; and a Liouville manifold can be decomposed along *sectorial hypersurfaces* into a union of sectorial pieces.

The objects of the (partially) wrapped Fukaya category of a Liouville sector [56] are properly embedded exact Lagrangian submanifolds which are cylindrical at infinity and stay away from the sectorial boundary; morphisms are given by homotopy colimits of Floer complexes for the images of the Lagrangians under increasingly large Hamiltonian perturbations in the Reeb direction at infinity (without crossing the sectorial boundary). The key result about the behavior of these categories is:

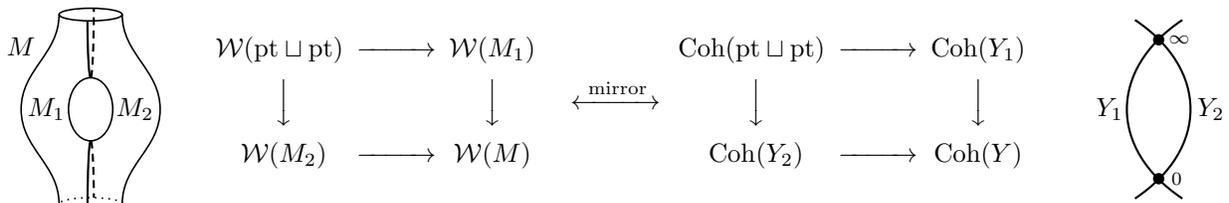
**THEOREM 3.1** (Ganatra-Pardon-Shende [56, 57]). *Wrapped Fukaya categories of Liouville sectors are covariantly functorial, i.e. a proper inclusion  $i : M' \hookrightarrow M$  gives rise to a functor  $i_* : \mathcal{W}(M') \rightarrow \mathcal{W}(M)$ . Moreover, when a Liouville manifold (or sector)  $M = M_1 \cup_{\partial} M_2$  is obtained by gluing two sectors  $M_1, M_2$  along the common sectorial boundary  $\partial M_i \simeq \mathbb{R} \times F$ , the inclusion functors give rise to a pushout diagram*

$$(3.2) \quad \begin{array}{ccc} \mathcal{W}(F) & \longrightarrow & \mathcal{W}(M_1) \\ \downarrow & & \downarrow \\ \mathcal{W}(M_2) & \longrightarrow & \mathcal{W}(M) \end{array}$$

(Note that  $\mathcal{W}(F)$  is equivalent to the wrapped Fukaya category of the *stabilized* sector  $T^*[0, 1] \times F$ , which admits proper inclusions to neighborhoods of the sectorial boundaries of  $M_1$  and  $M_2$ .) This allows one to compute  $\mathcal{W}(M)$  as the colimit of a diagram involving the inclusion functors from  $\mathcal{W}(F)$  into  $\mathcal{W}(M_1)$  and  $\mathcal{W}(M_2)$ . (Similarly for decompositions of a Liouville manifold or sector into more than two sectorial pieces.) See also [54] for further discussion of Liouville sectors and their Fukaya categories.

The gluing of wrapped Fukaya categories along sectorial decompositions typically corresponds under homological mirror symmetry to the gluing of categories of coherent sheaves under decompositions into unions of closed subschemes.

*Example 3.2.* A twice-punctured torus  $M$  can be cut along two arcs into a union of two sectors  $M_1, M_2$ , both isomorphic to a cylinder with two stops (boundary arcs). The corresponding inclusion functors give rise to a pushout diagram of wrapped Fukaya categories which, under homological mirror symmetry, matches the diagram of inclusion functors for coherent sheaves on the union of two copies of the projective line,  $Y_1, Y_2 \simeq \mathbb{P}^1$ , glued together at two points (say  $\{0, \infty\} \subset Y_i$ ), to produce a twice nodal elliptic curve  $Y = Y_1 \cup_{\{0, \infty\}} Y_2$ .



This perspective is particularly useful for studying mirror symmetry at the large limit, e.g. it can be used to understand Gammage and Shende's result on mirror symmetry for very affine hypersurfaces [50].

**Contravariant functoriality.** The above covariant functoriality notwithstanding, wrapped Fukaya categories generally behave in a contravariant manner with respect to inclusions of subdomains. The prototypical instance of this is the *restriction functor* constructed by Abouzaid and Seidel [12] from the wrapped Fukaya category of a Liouville manifold  $M$  to that of a *Liouville subdomain*  $M_- \subset M$  (i.e., a codimension 0 submanifold with convex boundary transverse to the Liouville vector field; the wrapped Fukaya category of  $M_-$  is defined to be that of its *completion*, the Liouville manifold  $\hat{M}_- = M_- \cup_{\partial M_-} [1, \infty) \times \partial M_-$ ). The construction relies on the same geometric idea as Viterbo’s earlier construction of a restriction map from the symplectic cohomology of  $M$  to that of  $M_-$  [103], namely the use of Hamiltonians which grow very steeply near  $\partial M_-$ , for which there are no Floer trajectories from generators in  $M \setminus M_-$  to generators in  $M_-$ , so that the chain-level restriction map amounts to a quotient by a subcomplex (in fact, an ideal with respect to the product operations).

Heather Lee’s thesis [68] used a similar idea to construct restriction functors associated to decompositions of Riemann surfaces into pairs of pants (overlapping in cylinders) and show that these give rise to pullback diagrams. (While Lee worked in the exact setting, the result also applies to Fukaya categories of closed surfaces [25].)

**THEOREM 3.3** (Lee [68]). *Given a decomposition of a Riemann surface  $\Sigma$  into a union of two subsurfaces  $\Sigma_1 \cup_\gamma \Sigma_2$  obtained by cutting  $\Sigma$  along a nontrivial simple closed curve (or union of such curves)  $\gamma$ , there are restriction functors from the wrapped Fukaya category of  $\Sigma$  to those of  $\Sigma_1, \Sigma_2$  and to a cylindrical neighborhood  $C$  of  $\gamma$ , which form a pullback diagram*

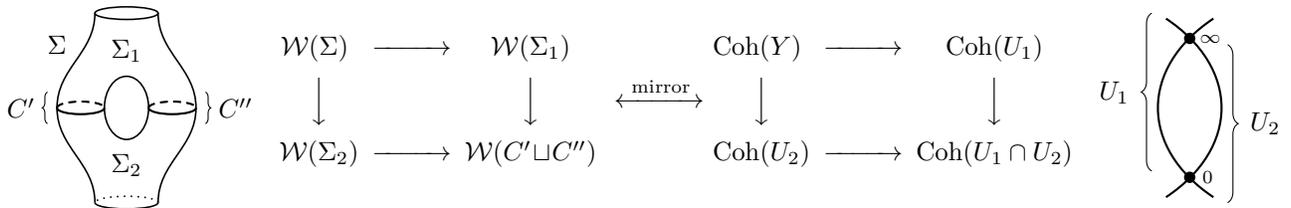
$$(3.3) \quad \begin{array}{ccc} \mathcal{W}(\Sigma) & \longrightarrow & \mathcal{W}(\Sigma_1) \\ \downarrow & & \downarrow \\ \mathcal{W}(\Sigma_2) & \longrightarrow & \mathcal{W}(C) \end{array}$$

(As before, by  $\mathcal{W}(\Sigma_i)$  and  $\mathcal{W}(C)$  we mean the wrapped Fukaya categories of the completions of these subsurfaces.)

Lee further showed that, for pair of pants decompositions of complex curves in  $(\mathbb{C}^*)^2$  near the tropical limit, the diagram (3.3) corresponds under mirror symmetry to the restriction functors associated to a natural affine cover of the mirror, i.e. the computation of wrapped Floer cohomology for a pair of Lagrangians in  $\Sigma$  via the pullback diagram matches the computation of sheaf cohomology via Čech complexes on the mirror [68].

See also [83] for a different perspective which leads to very similar results, but allows for further generalizations and applications to homological mirror symmetry in higher dimensions [74].

*Example 3.4.* As above, we consider a twice-punctured torus, but now we decompose it along two simple closed curves into two pairs of pants  $\Sigma_1, \Sigma_2$ , glued to each other along two cylindrical necks  $C', C''$ . The pair of pants is mirror to the affine curve  $\{xy = 0\} \subset \mathbb{A}^2$  (cf. e.g. [23, §3]), while the cylinder is mirror to the punctured affine line  $\mathbb{G}_m = \mathbb{A}^1 - \{0\}$ , and the restriction functors on wrapped Fukaya categories correspond under mirror symmetry to the restriction functors associated to the decomposition of  $Y = \mathbb{P}^1 \cup_{\{0, \infty\}} \mathbb{P}^1$  into two affine charts  $U_1, U_2$  (the complement of 0 and the complement of  $\infty$ ), with overlap  $U_1 \cap U_2 \simeq \mathbb{G}_m \sqcup \mathbb{G}_m$ .



It is important to note that Theorem 3.3 fails when one of the curves is homotopically trivial, i.e. one of the pieces of the decomposition is a disc. Indeed, in this case the locality of Floer trajectories for sufficiently steep Hamiltonians fails to hold: capping  $\Sigma_i$  with a disc deforms its Fukaya category nontrivially (cf. §2.4), and any local-to-global principle in such a setting must incorporate this deformation.

In the “closed-string” setting, an important conceptual step in this direction is the work of Groman and Varolgunes [60] defining the *relative symplectic cohomology*  $SH_M^*(V)$  of a domain  $V \subset M$ , which deforms  $SH^*(V)$  by the contributions of portions of Floer solutions in  $M \setminus V$  connecting generators in  $V$ . (When  $V$  is the complement of a divisor  $D$  in  $M$ , this amounts to the description of Floer theory in  $M$  as a deformation of Floer theory in  $M \setminus D$ , but the setting is much more general, allowing one to consider e.g. subsets of the form  $\pi^{-1}(U)$  for suitable

$U \subset B$  in the total space of a Lagrangian fibration  $\pi : M \rightarrow B$ .) One expects that a similar notion should exist for wrapped Fukaya categories and give rise to pullback diagrams of restriction functors associated to covers by subdomains, even in settings where the local categories have to be deformed by non-local contributions.

In the setting of SYZ mirror symmetry for a Lagrangian torus fibration  $\pi : M \rightarrow B$ , the relative symplectic cohomologies and relative wrapped Fukaya categories should behave like sheaves with respect to a cover of  $B$  by suitable domains. Returning to the discussion at the end of §3.1, the same should also be true of the corrected family Floer complex  $(\mathcal{C}, \delta + \{\mu^0, \cdot\})$  introduced in [24]; in fact the latter should be quasi-isomorphic (compatibly with  $BV$ -algebra structures) to the Čech complex of relative symplectic cohomologies, since both are expected to give Čech models for polyvector fields or Hochschild cochains on the corrected mirror.

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