

Fiber sums of genus 2 Lefschetz fibrations

Denis Auroux

ABSTRACT. Using the recent results of Siebert and Tian about the holomorphicity of genus 2 Lefschetz fibrations with irreducible singular fibers, we show that any genus 2 Lefschetz fibration becomes holomorphic after fiber sum with a holomorphic fibration.

1. Introduction

Symplectic Lefschetz fibrations have been the focus of a lot of attention since the proof by Donaldson that, after blow-ups, every compact symplectic manifold admits such structures [3]. Genus 2 Lefschetz fibrations, where the first non-trivial topological phenomena arise, have been particularly studied. Most importantly, it has recently been shown by Siebert and Tian that every genus 2 Lefschetz fibration without reducible fibers and with “transitive monodromy” is holomorphic [9]. The statement becomes false if reducible singular fibers are allowed, as evidenced by the construction by Ozbagci and Stipsicz [7] of genus 2 Lefschetz fibrations with non-complex total space (similar examples have also been constructed by Ivan Smith; the reader is also referred to the work of Amorós et al [1] and Korkmaz [5] for related constructions).

It has been conjectured by Siebert and Tian that any genus 2 Lefschetz fibration should become holomorphic after fiber sum with sufficiently many copies of the rational genus 2 Lefschetz fibration with 20 irreducible singular fibers. The purpose of this paper is to prove this conjecture by providing a classification of genus 2 Lefschetz fibrations up to stabilization by such fiber sums. The result is the following (see §2 and Definition 3.1 for notations):

Theorem 1.1. *Let F be any factorization of the identity element as a product of positive Dehn twists in the mapping class group Map_2 . Then there exist integers $\epsilon \in \{0, 1\}$, $k \geq 0$ and $m \geq 0$ such that, for any large enough integer n , the factorization $F \cdot (W_0)^n$ is Hurwitz equivalent to $(W_0)^{n+k} \cdot (W_1)^\epsilon \cdot (W_2)^m$.*

Corollary 1.2. *Let $f : X \rightarrow S^2$ be a genus 2 Lefschetz fibration. Then the fiber sum of f with sufficiently many copies of the rational genus 2 Lefschetz fibration with 20 irreducible singular fibers is isomorphic to a holomorphic fibration.*

2. Mapping class group factorizations

Recall that a Lefschetz fibration $f : X \rightarrow S^2$ is a fibration admitting only isolated singularities, all lying in distinct fibers of f , and near which a local model for f in

orientation-preserving complex coordinates is given by $(z_1, z_2) \mapsto z_1^2 + z_2^2$. We will only consider the case $\dim X = 4$, where the smooth fibers are compact oriented surfaces (of genus $g = 2$ in our case), and the singular fibers present nodal singularities obtained by collapsing a simple closed loop (the *vanishing cycle*) in the smooth fiber. The monodromy of the fibration around a singular fiber is given by a positive Dehn twist along the vanishing cycle.

Denoting by $q_1, \dots, q_r \in S^2$ the images of the singular fibers and choosing a reference point in S^2 , we can characterize the fibration f by means of its *monodromy* morphism $\psi : \pi_1(S^2 - \{q_1, \dots, q_r\}) \rightarrow \text{Map}_g$, where $\text{Map}_g = \pi_0 \text{Diff}^+(\Sigma_g)$ is the mapping class group of an oriented genus g surface. It is a classical result (cf. [4]) that the monodromy morphism ψ is uniquely determined up to conjugation by an element of Map_g and a braid acting on $\pi_1(S^2 - \{q_i\})$, and that it determines the isomorphism class of the Lefschetz fibration f .

While all positive Dehn twists along non-separating curves are mutually conjugate in Map_g , there are different types of twists along separating curves, according to the genus of each component delimited by the curve. When $g = 2$, only two cases can occur: either the curve splits the surface into two genus 1 components, or it is homotopically trivial and the corresponding singular fiber contains a sphere component of square -1 . The latter case can always be avoided by blowing down the total space of the fibration; if the blown-down fibration can be shown to be holomorphic, then by performing the converse blow-up procedure we conclude that the original fibration was also holomorphic. Therefore, in all the following we can assume that our Lefschetz fibrations are *relatively minimal*, i.e. have no homotopically trivial vanishing cycles.

The monodromy of a Lefschetz fibration can be encoded in a *mapping class group factorization* by choosing an ordered system of generating loops $\gamma_1, \dots, \gamma_r$ for $\pi_1(S^2 - \{q_i\})$, such that each loop γ_i encircles only one of the points q_i and $\prod \gamma_i$ is homotopically trivial. The monodromy of the fibration along each of the loops γ_i is a Dehn twist τ_i ; we can then describe the fibration in terms of the relation $\tau_1 \cdot \dots \cdot \tau_r = 1$ in Map_2 . The choice of the loops γ_i (and therefore of the twists τ_i) is of course not unique, but any two choices differ by a sequence of *Hurwitz moves* exchanging consecutive factors: $\tau_i \cdot \tau_{i+1} \rightarrow (\tau_{i+1})_{\tau_i}^{-1} \cdot \tau_i$ or $\tau_i \cdot \tau_{i+1} \rightarrow \tau_{i+1} \cdot (\tau_i)_{\tau_{i+1}}$, where we use the notation $(\tau)_\phi = \phi^{-1} \tau \phi$, i.e. if τ is a Dehn twist along a loop δ then $(\tau)_\phi$ is the Dehn twist along the loop $\phi(\delta)$.

Definition 2.1. A *factorization* $F = \tau_1 \cdot \dots \cdot \tau_r$ in Map_g is an ordered tuple of positive Dehn twists. We say that two factorizations are *Hurwitz equivalent* ($F \sim F'$) if they can be obtained from each other by a sequence of Hurwitz moves.

It is well-known that a Lefschetz fibration is characterized by a factorization of the identity element in Map_g , uniquely determined up to Hurwitz equivalence and simultaneous conjugation of all factors by a same element of Map_g .

Let ζ_i ($1 \leq i \leq 5$) and σ be the Dehn twists represented in Figure 1. It is well-known (cf. e.g. [2], Theorem 4.8) that Map_2 admits the following presentation:

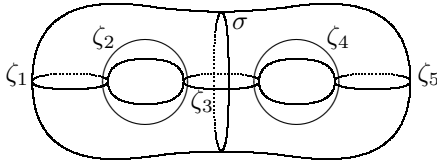


FIGURE 1

- generators: ζ_1, \dots, ζ_5 .
- relations: $\zeta_i \zeta_j = \zeta_j \zeta_i$ if $|i - j| \geq 2$; $\zeta_i \zeta_{i+1} \zeta_i = \zeta_{i+1} \zeta_i \zeta_{i+1}$;
 $(\zeta_1 \zeta_2 \zeta_3 \zeta_4 \zeta_5)^6 = 1$; $I = \zeta_1 \zeta_2 \zeta_3 \zeta_4 \zeta_5^2 \zeta_4 \zeta_3 \zeta_2 \zeta_1$ is central; $I^2 = 1$.

It is easy to check that σ can be expressed in terms of the generators ζ_1, \dots, ζ_5 as $\sigma = (\zeta_1 \zeta_2)^6 = (\zeta_4 \zeta_5)^6 = (\zeta_1 \zeta_2)^3 (\zeta_4 \zeta_5)^3 I$.

We can fix a hyperelliptic structure on the genus 2 surface Σ , i.e. a double covering map $\Sigma \rightarrow S^2$ (with 6 branch points), in such a way that ζ_1, \dots, ζ_5 become the lifts of standard half-twists exchanging consecutive branch points in S^2 . The element I then corresponds to the hyperelliptic involution (i.e. the non-trivial automorphism of the double covering). The fact that I is central means that every diffeomorphism of Σ is compatible with the hyperelliptic structure, up to isotopy. In fact, Map_2 is closely related to the braid group $B_6(S^2)$ acting on the branch points of the double covering. The group $B_6(S^2)$ admits the following presentation (cf. [2], Theorem 1.1):

- generators: x_1, \dots, x_5 (half-twists exchanging two consecutive points).
- relations: $x_i x_j = x_j x_i$ if $|i - j| \geq 2$; $x_i x_{i+1} x_i = x_{i+1} x_i x_{i+1}$;
 $x_1 x_2 x_3 x_4 x_5^2 x_4 x_3 x_2 x_1 = 1$.

Consider a S^2 -bundle $\pi : P \rightarrow S^2$, and a smooth curve $B \subset P$ intersecting a generic fiber in 6 points, everywhere transverse to the fibers of π except for isolated nondegenerate complex tangencies. The curve B can be characterized by its *braid monodromy*, or equivalently by a factorization in the braid group $B_6(S^2)$, with each factor a positive half-twist, defined by considering the motion of the 6 intersection points of B with the fiber of π upon moving around the image of a tangency point. As before, this factorization is only defined up to Hurwitz equivalence and simultaneous conjugation (see also [6] for the case of plane curves).

There exists a lifting morphism from $B_6(S^2)$ to $\text{Map}_2/\langle I \rangle$, defined by $x_i \mapsto \zeta_i$. Given a half-twist in $B_6(S^2)$, exactly one of its two possible lifts to Map_2 is a Dehn twist about a non-separating curve. This allows us to lift the braid factorization associated to the curve $B \subset P$ to a mapping class group factorization; the product of the resulting factors is equal to 1 if the homology class represented by B is divisible by two, and to I otherwise. In the first case, we can construct a genus 2 Lefschetz fibration by considering the double covering of P branched along B , and its monodromy is exactly the lift of the braid monodromy of the curve B . This construction always yields Lefschetz fibrations without reducible singular fibers; however, if we additionally allow some blow-up and blow-down

operations (on P and its double covering respectively), then we can also handle the case of reducible singular fibers (see §3 below and [8]). It is worth mentioning that Siebert and Tian have shown the converse result: given any genus 2 Lefschetz fibration, it can be realized as a double covering of a S^2 -bundle over S^2 (with additional blow-up and blow-down operations in the case of reducible singular fibers) [8].

3. Holomorphic genus 2 fibrations

We are interested in the properties of certain specific factorizations in Map_2 .

Definition 3.1. Let $W_0 = (T)^2$, $W_1 = (\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5)^6$, and $W_2 = \sigma \cdot (\zeta_3 \cdot \zeta_4 \cdot \zeta_5 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_1 \cdot \zeta_2 \cdot \zeta_3)^2 \cdot (T)$, where $T = \zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5 \cdot \zeta_5 \cdot \zeta_4 \cdot \zeta_3 \cdot \zeta_2 \cdot \zeta_1$.

In this definition the notation $(\dots)^n$ means that the sequence of Dehn twists is repeated n times. It is fairly easy to check that W_0 , W_1 and W_2 are all factorizations of the identity element of Map_2 as a product of 20, 30, and 29 positive Dehn twists respectively (for W_0 and W_1 this follows immediately from the presentation of Map_2 ; see below for W_2).

Lemma 3.1. *The factorization W_0 describes the genus 2 Lefschetz fibration f_0 on the rational surface obtained as a double covering of $\mathbb{F}_0 = \mathbb{C}\mathbb{P}^1 \times \mathbb{C}\mathbb{P}^1$ branched along a smooth algebraic curve B_0 of bidegree $(6, 2)$. The factorization W_1 corresponds to the genus 2 Lefschetz fibration f_1 on the blown-up $K3$ surface obtained as a double covering of $\mathbb{F}_1 = \mathbb{C}\mathbb{P}^2 \# \mathbb{C}\mathbb{P}^2$ branched along a smooth algebraic curve B_1 in the linear system $|6L|$, where L is a line in $\mathbb{C}\mathbb{P}^2$ avoiding the blown-up point.*

Proof. B_0 can be degenerated into a singular curve D_0 consisting of 6 sections and 2 fibers intersecting in 12 nodes (see Figure 2). We can recover B_0 from D_0 by first smoothing the intersections of the first section with the two fibers, giving us a component of bidegree $(1, 2)$, and then smoothing the remaining 10 nodes, each of which produces two vertical tangencies. The braid factorization corresponding to B_0 can therefore be expressed as $((x_1)^2 \cdot (x_2)_{x_1}^2 \cdot (x_3)_{x_2 x_1}^2 \cdot (x_4)_{x_3 x_2 x_1}^2 \cdot (x_5)_{x_4 x_3 x_2 x_1}^2)^2$, or equivalently after suitable Hurwitz moves, $(x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_5 \cdot x_4 \cdot x_3 \cdot x_2 \cdot x_1)^2$. Lifting this braid factorization to the mapping class group, we obtain W_0 as claimed. Alternately, it is easy to check that the braid factorization for a smooth curve of bidegree $(6, 1)$ is $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_5 \cdot x_4 \cdot x_3 \cdot x_2 \cdot x_1$; we can then conclude by observing that B_0 is the fiber sum of two such curves.

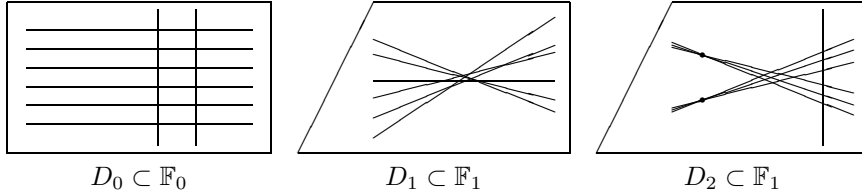


FIGURE 2

In the case of the curve B_1 , by definition the braid monodromy is exactly that of a smooth plane curve of degree 6 as defined by Moishezon in [6]; it can be computed e.g. by degenerating B_1 to a union of 6 lines in generic position (D_1 in Figure 2), and is known to be given by the factorization $(x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5)^6$. Lifting to Map_2 , we obtain that the monodromy factorization for the corresponding double branched covering is exactly W_1 . \square

Lemma 3.2. *Let $\tau \in \text{Map}_2$ be a Dehn twist, and let F be a factorization of a central element (1 or I) of Map_2 . If $F \sim \tau \cdot F'$ for some F' , then the factorization $(F)_\tau$ obtained from F by simultaneous conjugation of all factors by τ is Hurwitz equivalent to F .*

Proof. We have: $(F)_\tau \sim \tau \cdot (F')_\tau \sim F' \cdot \tau \sim (\tau)_{F'} \cdot F' = \tau \cdot F' \sim F$. The first and last steps follow from the assumption; the second step corresponds to moving τ to the right across all the factors of $(F')_\tau$, while in the third step all the factors of F' are moved to the right across τ . Also observe that $(\tau)_{F'} = \tau$ because the product of all factors in F' commutes with τ . \square

Lemma 3.3. *The factorizations T , W_0 and W_1 are fully invariant, i.e. for any element $\gamma \in \text{Map}_2$ we have $(T)_\gamma \sim T$, $(W_0)_\gamma \sim W_0$, and $(W_1)_\gamma \sim W_1$.*

Proof. It is obviously sufficient to prove that $(T)_{\zeta_i} \sim T$ and $(W_1)_{\zeta_i} \sim W_1$ for all $1 \leq i \leq 5$. By moving the first ζ_i factor in T or W_1 to the left, we obtain a Hurwitz equivalent factorization of the form $\zeta_i \dots$; therefore the result follows immediately from Lemma 3.2. \square

A direct consequence of Lemma 3.3 is that all fiber sums of the holomorphic fibrations f_0 and f_1 (with monodromies W_0 and W_1) are *untwisted*. More precisely, when two Lefschetz fibrations with monodromy factorizations F and F' are glued to each other along a fiber, the resulting fibration normally depends on the isotopy class ϕ of a diffeomorphism between the two fibers to be identified, and its monodromy is given by a factorization of the form $(F) \cdot (F')_\phi$. However, when the building blocks are made of copies of f_0 and f_1 , Lemma 3.3 implies that the result of the fiber sum operation is independent of the chosen identification diffeomorphisms; e.g., we can always take $\phi = 1$.

Lemma 3.4. $(W_1)^2 \sim (W_0)^3$.

Proof. Let $\rho = \zeta_1 \zeta_2 \zeta_3 \zeta_4 \zeta_5 \zeta_1 \zeta_2 \zeta_3 \zeta_4 \zeta_1 \zeta_2 \zeta_3 \zeta_1 \zeta_2 \zeta_1$ be the reflection of the genus 2 surface Σ about its central axis. It follows from Lemma 3.3 that $(W_1)^2 \sim W_1 \cdot (W_1)_\rho = (\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5)^6 \cdot (\zeta_5 \cdot \zeta_4 \cdot \zeta_3 \cdot \zeta_2 \cdot \zeta_1)^6$. The central part of this factorization is exactly T ; after moving it to the right, we obtain the new identity $(W_1)^2 \sim (\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5)^5 \cdot (\zeta_5 \cdot \zeta_4 \cdot \zeta_3 \cdot \zeta_2 \cdot \zeta_1)^5 \cdot (T)_{(\zeta_5 \zeta_4 \zeta_3 \zeta_2 \zeta_1)^5}$. Repeating the same operation four more times, we get $(W_1)^2 \sim \prod_{j=0}^5 (T)_{(\zeta_5 \zeta_4 \zeta_3 \zeta_2 \zeta_1)^j}$. Using the full invariance property of T (Lemma 3.3), it follows that $(W_1)^2 \sim (T)^6 = (W_0)^3$.

A more geometric argument is as follows: $(W_1)^2$ is the monodromy factorization of the fiber sum $f_1 \# f_1$ of two copies of f_1 , which is a double covering of the fiber sum

of two copies of (\mathbb{F}_1, B_1) . Therefore $f_1 \# f_1$ is a double covering of (\mathbb{F}_2, B') where $\mathbb{F}_2 = \mathbb{P}(O \oplus O(2))$ is the second Hirzebruch surface and B' is a smooth algebraic curve in the linear system $|6S|$, where S is a section of \mathbb{F}_2 ($S \cdot S = 2$). On the other hand $(W_0)^3$ is the monodromy factorization of the fiber sum $f_0 \# f_0 \# f_0$, which is a double covering of the fiber sum of three copies of (\mathbb{F}_0, B_0) , i.e. a double covering of (\mathbb{F}_0, B'') where B'' is a smooth algebraic curve of bidegree $(6, 6)$. The conclusion follows from the fact that (\mathbb{F}_2, B') and (\mathbb{F}_0, B'') are deformation equivalent. \square

We can now reformulate the holomorphicity result obtained by Siebert and Tian [9] in terms of mapping class group factorizations. Say that a factorization is *transitive* if the images of the factors under the morphism $\text{Map}_2 \rightarrow S_6$ mapping ζ_i to the transposition $(i, i + 1)$ generate the entire symmetric group S_6 .

Theorem 3.5 (Siebert-Tian [9]). *Any transitive factorization of the identity element as a product of positive Dehn twists along non-separating curves in Map_2 is Hurwitz equivalent to a factorization of the form $(W_0)^k \cdot (W_1)^\epsilon$ for some integers $k \geq 0$ and $\epsilon \in \{0, 1\}$.*

What Siebert and Tian have shown is in fact that any such factorization is the monodromy of a holomorphic Lefschetz fibration, which can be realized as a double covering of a ruled surface branched along a smooth connected holomorphic curve intersecting the generic fiber in 6 points. However, we can always assume that the ruled surface is either \mathbb{F}_0 or \mathbb{F}_1 (either by the topological classification of ruled surfaces or using Lemma 3.4). In the first case, the branch curve has bidegree $(6, 2k)$ for some integer k , and the corresponding monodromy is $(W_0)^k$, while in the second case the branch curve realizes the homology class $6[L] + 2k[F]$ for some integer k (here F is a fiber of \mathbb{F}_1), and the corresponding monodromy is $(W_0)^k \cdot W_1$.

We now look at examples of genus 2 fibrations with reducible singular fibers.

Definition 3.2. Let $B_2 \subset \mathbb{F}_1$ be an algebraic curve in the linear system $|6L + F|$, presenting two triple points in the same fiber F_0 . Let P_2 be the surface obtained by blowing up \mathbb{F}_1 at the two triple points of B_2 , and denote by \hat{B}_2 and \hat{F}_0 the proper transforms of B_2 and F_0 in P_2 . Consider the double covering $\pi : \hat{X}_2 \rightarrow P_2$ branched along $\hat{B}_2 \cup \hat{F}_0$, and let X_2 be the surface obtained by blowing down the -1 -curve $\pi^{-1}(\hat{F}_0)$ in \hat{X}_2 .

Let us check that this construction is well-defined. The easiest way to construct the curve B_2 is to start with a curve C of degree 7 in $\mathbb{C}\mathbb{P}^2$ with two triple points p_1 and p_2 . If we choose C generically, we can assume that the three branches of C through p_i intersect each other transversely and are transverse to the line L_0 through p_1 and p_2 . Therefore the line L_0 intersects C transversely in another point p , and by blowing up $\mathbb{C}\mathbb{P}^2$ at p we obtain the desired curve B_2 (see also below). Next, we blow up the two triple points p_1 and p_2 , which turns B_2 into a smooth curve \hat{B}_2 , disjoint from \hat{F}_0 . Denoting by E_1 and E_2 the exceptional divisors of the two blow-ups, we have $[\hat{B}_2] = 6[L] + [F] - 3[E_1] - 3[E_2]$ and $[\hat{F}_0] = [F] - [E_1] - [E_2]$, so that $[\hat{B}_2] + [\hat{F}_0] = 6[L] + 2[F] - 4[E_1] - 4[E_2]$ is divisible by 2; therefore the double covering $\pi : \hat{X}_2 \rightarrow P_2$ is well-defined.

The complex surface \hat{X}_2 is equipped with a natural holomorphic genus 2 fibration \hat{f}_2 , obtained by composing $\pi : \hat{X}_2 \rightarrow P_2$ with the natural projections to \mathbb{F}_1 and then to S^2 . The fiber of \hat{f}_2 corresponding to $F_0 \subset \mathbb{F}_1$ consists of three components: two elliptic curves of square -2 obtained as double coverings of the exceptional curves E_1 and E_2 in P_2 , with 4 branch points in each case (three on \hat{B}_2 and one on \hat{F}_0), and a rational curve of square -1 , the preimage of \hat{F}_0 . After blowing down the rational component, we obtain on X_2 a holomorphic genus 2 fibration f_2 , with one reducible fiber consisting of two elliptic components. It is easy to check that near this singular point f_2 presents the local model expected of a Lefschetz fibration, and that the vanishing cycle for this fiber is the loop obtained by lifting any simple closed loop that separates the two triple points of B_2 inside the fiber F_0 of \mathbb{F}_1 .

Lemma 3.6. *The complex surface X_2 carries a natural holomorphic genus 2 Lefschetz fibration, with monodromy factorization W_2 .*

Proof. We need to calculate the braid monodromy factorization associated to the curve $B_2 \subset \mathbb{F}_1$. For this purpose, observe that B_2 can be degenerated to a union D_2 of 6 lines in two groups of three, L_1, L_2, L_3 and L_4, L_5, L_6 , and a fiber F , with two triple points and 15 nodes (cf. Figure 2). The monodromy around the fiber containing the two triple points is given by the braid $\delta = (x_1 x_2)^3 (x_4 x_5)^3$. The 9 nodes corresponding to the mutual intersections of the two groups of three lines give rise to 18 vertical tangencies in B_2 , and the corresponding factorization is

$$x_3^2 \cdot (x_4)_{x_3}^2 \cdot (x_5)_{x_4 x_3}^2 \cdot [x_2^2 \cdot (x_3)_{x_2}^2 \cdot (x_4)_{x_3 x_2}^2]_{(x_5 x_4 x_3)} \cdot [x_1^2 \cdot (x_2)_{x_1}^2 \cdot (x_3)_{x_2 x_1}^2]_{(x_4 x_3 x_2 x_5 x_4 x_3)}.$$

After suitable Hurwitz moves, this expression can be rewritten as

$$x_3 \cdot x_4 \cdot (x_5)^2 \cdot x_4 \cdot x_3 \cdot [x_2 \cdot x_3 \cdot (x_4)^2 \cdot x_3 \cdot x_2]_{(x_5 x_4 x_3)} \cdot [x_1 \cdot x_2 \cdot (x_3)^2 \cdot x_2 \cdot x_1]_{(x_4 x_3 x_2 x_5 x_4 x_3)},$$

or equivalently as $x_3 \cdot x_4 \cdot x_5 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_1 \cdot x_2 \cdot x_3 \cdot x_3 \cdot x_2 \cdot x_1 \cdot x_4 \cdot x_3 \cdot x_2 \cdot x_5 \cdot x_4 \cdot x_3$, which is in turn equivalent to $(x_3 \cdot x_4 \cdot x_5 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_1 \cdot x_2 \cdot x_3)^2$. Finally, the six intersections of the lines L_1, \dots, L_6 with the fiber F give rise to 10 vertical tangencies, for which the same argument as for Lemma 3.1 gives the monodromy factorization $x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 \cdot x_5 \cdot x_4 \cdot x_3 \cdot x_2 \cdot x_1$. We conclude by lifting the monodromy of B_2 to the mapping class group, observing that the contribution δ of the fiber containing the triple points lifts to the Dehn twist σ . \square

Theorem 3.7. *Fix integers $m \geq 0$, $\epsilon \in \{0, 1\}$ and $k \geq \frac{3}{2}m + 1$. Then the Hirzebruch surface $\mathbb{F}_{m+\epsilon} = \mathbb{P}(O \oplus O(m+\epsilon))$ contains a complex curve $B_{k,\epsilon,m}$ in the linear system $|6S + (m+2k)F|$ (where S is a section of square $(m+\epsilon)$ and F is a fiber), having $2m$ triple points lying in m distinct fibers of $\mathbb{F}_{m+\epsilon}$ as its only singularities.*

Moreover, after blowing up $\mathbb{F}_{m+\epsilon}$ at the $2m$ triple points, passing to a double covering, and blowing down m rational -1 -curves, we obtain a complex surface and a holomorphic genus 2 fibration $f_{k,\epsilon,m} : X_{k,\epsilon,m} \rightarrow S^2$ with monodromy factorization $(W_0)^k \cdot (W_1)^\epsilon \cdot (W_2)^m$.

Proof. We first construct the curve $B_{k,\epsilon,m}$ by perturbation of a singular configuration $D_{k,\epsilon,m}$ consisting of 6 sections of $\mathbb{F}_{m+\epsilon}$ together with $m+2k$ fibers. Since the case of

smooth curves is a classical result, we can assume that $m \geq 1$. Also observe that, since the intersection number of $B_{k,\epsilon,m}$ with a fiber is equal to 6, the $2m$ triple points must come in pairs lying in the same fiber: $p_{2i-1}, p_{2i} \in F_i$, $1 \leq i \leq m$.

Let u_0 and u_1 be generic sections of the line bundle $O_{\mathbb{CP}^1}(m+\epsilon)$, without common zeroes. Define six sections $S_{\alpha,\beta}$ ($\alpha \in \{0,1\}$, $\beta \in \{0,1,2\}$) of $\mathbb{F}_{m+\epsilon}$ as the projectivizations of the sections $(1, (-1)^\alpha u_0 + c_{\alpha\beta} u_1)$ of $O_{\mathbb{CP}^1} \oplus O_{\mathbb{CP}^1}(m+\epsilon)$, where $c_{\alpha\beta}$ are small generic complex numbers. The three sections $S_{0,\beta}$ intersect each other in $(m+\epsilon)$ triple points, in the fibers $F_1, \dots, F_{m+\epsilon}$ above the points of \mathbb{CP}^1 where u_1 vanishes, and similarly for the three sections $S_{1,\beta}$; generic choices of parameters ensure that all the other intersections between these six sections are transverse and lie in different fibers of $\mathbb{F}_{m+\epsilon}$. We define $D_{k,\epsilon,m}$ to be the singular configuration consisting of the six sections $S_{\alpha,\beta}$ together with $m+2k$ generic fibers of $\mathbb{F}_{m+\epsilon}$ intersecting the sections in six distinct points.

Let $s \in H^0(\mathbb{F}_{m+\epsilon}, O(mF))$ be the product of the sections of $O(F)$ defining the fibers F_1, \dots, F_m containing $2m$ of the triple points of $D_{k,\epsilon,m}$. Let s' be a generic section of the line bundle $L = O(D_{k,\epsilon,m} - 4mF) = O(6S + (2k-3m)F)$ over $\mathbb{F}_{m+\epsilon}$. Because $k \geq \frac{3}{2}m+1$, the linear system $|L|$ is base point free, and so we can assume that s' does not vanish at any of the double or triple points of $D_{k,\epsilon,m}$. Finally, let s_0 be the section defining $D_{k,\epsilon,m}$.

We consider the section $s_\lambda = s_0 + \lambda s^4 s' \in H^0(\mathbb{F}_{m+\epsilon}, O(6S + (2k+m)F))$, where $\lambda \neq 0$ is a generic small complex number. Because the perturbation $s^4 s'$ vanishes at order 4 at each of the triple points p_1, \dots, p_{2m} of $D_{k,\epsilon,m}$ in the fibers F_1, \dots, F_m , it is easy to check that all the curves $D(\lambda) = s_\lambda^{-1}(0)$ present triple points at p_1, \dots, p_{2m} . On the other hand, since $s^4 s'$ does not vanish at any of the other singular points of $D_{k,\epsilon,m}$, for generic λ the curve $D(\lambda)$ presents no other singularities than the triple points p_1, \dots, p_{2m} (this follows e.g. from Bertini's theorem); this gives us the curve $B_{k,\epsilon,m}$ with the desired properties. Moreover, generic choices of the parameters ensure that the vertical tangencies of $B_{k,\epsilon,m}$ all lie in distinct fibers of $\mathbb{F}_{m+\epsilon}$; in that case, the double covering construction will give rise to a Lefschetz fibration.

The braid monodromy of the curve $B_{k,\epsilon,m}$ can be computed using the existence of a degeneration to the singular configuration $D_{k,\epsilon,m}$ (taking $\lambda \rightarrow 0$ in the above construction); a calculation similar to the proofs of Lemma 3.1 and Lemma 3.6 yields that it consists of k copies of the braid factorization of the curve B_0 , ϵ copies of the braid factorization of B_1 , and m copies of the braid factorization of B_2 . Another way to see this is to observe that the surface $\mathbb{F}_{m+\epsilon}$ admits a decomposition into a fiber sum of k copies of \mathbb{F}_0 and $m+\epsilon$ copies of \mathbb{F}_1 , in such a way that the singular configuration $D_{k,\epsilon,m}$ naturally decomposes into k copies of D_0 , ϵ copies of a degenerate version of D_1 presenting some triple points, and m copies of D_2 . After a suitable smoothing, we obtain that the pair $(\mathbb{F}_{m+\epsilon}, B_{k,\epsilon,m})$ splits as the untwisted fiber sum $k(\mathbb{F}_0, B_0) \#_\epsilon (\mathbb{F}_1, B_1) \# m(\mathbb{F}_1, B_2)$.

By the same process as in the construction of the surface X_2 , we can blow up the $2m$ triple points of $B_{k,\epsilon,m}$, take a double covering branched along the proper transforms of $B_{k,\epsilon,m}$ and of the fibers through the triple points, and blow down m rational components, to obtain a complex surface $X_{k,\epsilon,m}$ equipped with a holomorphic genus 2 Lefschetz fibration $f_{k,\epsilon,m} : X_{k,\epsilon,m} \rightarrow S^2$. Because of the structure of $B_{k,\epsilon,m}$, it is easy to see that $f_{k,\epsilon,m}$

splits into an untwisted fiber sum of k copies of f_0 , ϵ copies of f_1 , and m copies of f_2 . Therefore, its monodromy is described by the factorization $(W_0)^k \cdot (W_1)^\epsilon \cdot (W_2)^m$. \square

4. Proof of the main result

In order to prove Theorem 1.1, we will use the following lemma which allows us to trade one reducible singular fiber against a collection of irreducible singular fibers:

Lemma 4.1. $(\zeta_1 \cdot \zeta_2)^3 \cdot (\zeta_4 \cdot \zeta_5)^3 \cdot (T) \cdot (W_2) \sim \sigma \cdot (W_0) \cdot (W_1)$.

Proof. Let $\Phi = \zeta_3 \cdot \zeta_4 \cdot \zeta_5 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_1 \cdot \zeta_2 \cdot \zeta_3$, and observe that $(\zeta_i)_\Phi = \zeta_{6-i}$ for $i \in \{1, 2, 4, 5\}$. Therefore, we have

$$(\zeta_4 \cdot \zeta_5)^3 \cdot (\Phi)^2 \sim (\Phi) \cdot \zeta_1 \cdot \zeta_2 \cdot \zeta_1 \cdot (\Phi) \cdot \zeta_5 \cdot \zeta_4 \cdot \zeta_5 \sim (\Phi) \cdot (\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5)^3.$$

Moreover, $(\zeta_1 \cdot \zeta_2)^3 \cdot (\Phi) \sim \zeta_1 \cdot \zeta_2 \cdot \zeta_1 \cdot (\Phi) \cdot \zeta_5 \cdot \zeta_4 \cdot \zeta_5 \sim (\zeta_1 \cdot \zeta_2 \cdot \zeta_3 \cdot \zeta_4 \cdot \zeta_5)^3$. It follows that $(\zeta_1 \cdot \zeta_2)^3 \cdot (\zeta_4 \cdot \zeta_5)^3 \cdot (\Phi)^2 \sim W_1$. Recalling that $(\zeta_1 \zeta_2)^3 (\zeta_4 \zeta_5)^3 = \sigma I$ and $W_2 = \sigma \cdot (\Phi)^2 \cdot (T)$, and using the invariance property of T (Lemma 3.3), we have

$$(\zeta_1 \cdot \zeta_2)^3 \cdot (\zeta_4 \cdot \zeta_5)^3 \cdot (T) \cdot (W_2) \sim \sigma \cdot (T) \cdot (\zeta_1 \cdot \zeta_2)^3 \cdot (\zeta_4 \cdot \zeta_5)^3 \cdot (\Phi)^2 \cdot (T) \sim \sigma \cdot (T) \cdot (W_1) \cdot (T),$$

which is Hurwitz equivalent to $\sigma \cdot (T)^2 \cdot (W_1) = \sigma \cdot (W_0) \cdot (W_1)$. \square

Proof of Theorem 1.1. We argue by induction on the number m of reducible singular fibers. If there are no separating Dehn twists, then after summing with at least one copy of W_0 to ensure transitivity, we obtain a transitive factorization of the identity element into Dehn twists along non-separating curves, which by Theorem 3.5 is of the expected form.

Assume that Theorem 1.1 holds for all factorizations with $m-1$ separating Dehn twists, and consider a factorization F with m separating Dehn twists. By Hurwitz moves we can bring one of the separating Dehn twists to the right-most position in F and assume that $F = (F') \cdot \tilde{\sigma}$, where $\tilde{\sigma}$ is a Dehn twist about a loop separating two genus 1 components. Clearly, there exists an element $\phi \in \text{Map}_2$ such that $\tilde{\sigma} = (\sigma)_{\phi^{-1}}$. Using the relation $I^2 = 1$, we can express each ζ_i^{-1} as a product of the generators ζ_1, \dots, ζ_5 , and therefore ϕ can be expressed as a positive word (ϕ) involving only the generators ζ_1, \dots, ζ_5 (and not their inverses).

Since all five generators appear in the expression of W_0 , for large enough n we can perform Hurwitz moves on $(W_0)^n$ in order to collect the factors of (ϕ) to the left and obtain that $(W_0)^n \sim (\phi) \cdot W'$, for some factorization W' . Starting with the factorization $\tilde{\sigma} \cdot (W_0)^n$, we can move $\tilde{\sigma}$ to the right across the various factors: $\tilde{\sigma} \cdot (W_0)^n \sim (\sigma)_{\phi^{-1}} \cdot (\phi) \cdot W' \sim (\phi) \cdot \sigma \cdot W' \sim (\phi) \cdot (W')_{\sigma^{-1}} \cdot \sigma$. Consequently, if n is sufficiently large, we obtain that $\tilde{\sigma} \cdot (W_0)^n \sim (F'') \cdot \sigma$, for some factorization F'' involving only non-separating Dehn twists. Therefore, using Lemma 3.4 and Lemma 4.1 we have

$$F \cdot (W_0)^{n+4} \sim F' \cdot \tilde{\sigma} \cdot (W_0)^n \cdot W_0 \cdot (W_1)^2 \sim F' \cdot F'' \cdot \sigma \cdot W_0 \cdot (W_1)^2 \sim \tilde{F} \cdot W_2,$$

where $\tilde{F} = F' \cdot F'' \cdot (\zeta_1 \cdot \zeta_2)^3 \cdot (\zeta_4 \cdot \zeta_5)^3 \cdot T \cdot W_1$. Next we observe that \tilde{F} is a factorization of the identity element with $m-1$ separating Dehn twists, therefore by assumption there

exist integers \tilde{n}, k, ϵ such that $\tilde{F} \cdot (W_0)^{\tilde{n}} \sim (W_0)^k \cdot (W_1)^\epsilon \cdot (W_2)^{m-1}$. It follows that $F \cdot (W_0)^{n+\tilde{n}+4} \sim \tilde{F} \cdot W_2 \cdot (W_0)^{\tilde{n}} \sim \tilde{F} \cdot (W_0)^{\tilde{n}} \cdot W_2 \sim (W_0)^k \cdot (W_1)^\epsilon \cdot (W_2)^m$. This concludes the proof, since it is clear that the splitting remains valid after adding extra copies of W_0 . \square

Proof of Corollary 1.2. First of all, as observed at the beginning of §2 we can assume that f is relatively minimal, i.e. all vanishing cycles are homotopically non-trivial. Let F be a monodromy factorization corresponding to f , and observe that by Theorem 1.1 we have a splitting of the form $F \cdot (W_0)^n \sim (W_0)^{n+k} \cdot (W_1)^\epsilon \cdot (W_2)^m$. If n is chosen large enough then $n+k \geq \frac{3}{2}m+1$, and so by Theorem 3.7 the right-hand side is the monodromy of the holomorphic fibration $f_{n+k, \epsilon, m}$, while the left-hand side corresponds to the fiber sum $f \# n f_0$. \square

Remark. Pending a suitable extension of the result of Siebert and Tian to higher genus hyperelliptic Lefschetz fibrations with transitive monodromy and irreducible singular fibers, the techniques described here can be generalized to higher genus hyperelliptic fibrations almost without modification. The main difference is the existence of different types of reducible fibers, classified by the genera h and $g-h$ of the two components; this makes it necessary to replace W_2 with a larger collection of building blocks, obtained e.g. from complex curves in \mathbb{F}_1 that intersect the generic fiber in $2g+2$ points and present two multiple points with multiplicities $2h+1$ and $2(g-h)+1$ in the same fiber.

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DEPARTMENT OF MATHEMATICS, M.I.T., CAMBRIDGE MA 02139, USA
E-mail address: `auroux@math.mit.edu`

CENTRE DE MATHÉMATIQUES, ECOLE POLYTECHNIQUE, F-91128 PALAISEAU CEDEX, FRANCE
E-mail address: `auroux@math.polytechnique.fr`