

Fukaya categories and bordered Heegaard-Floer homology

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Abstract. We outline an interpretation of Heegaard-Floer homology of 3-manifolds (closed or with boundary) in terms of the symplectic topology of symmetric products of Riemann surfaces, as suggested by recent work of Tim Perutz and Yankı Lekili. In particular we discuss the connection between the Fukaya category of the symmetric product and the bordered algebra introduced by Robert Lipshitz, Peter Ozsváth and Dylan Thurston, and recast bordered Heegaard-Floer homology in this language.

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1. Introduction

In its simplest incarnation, Heegaard-Floer homology associates to a closed 3-manifold Y a graded abelian group $\widehat{HF}(Y)$. This invariant is constructed by considering a Heegaard splitting $Y = Y_1 \cup_{\Sigma} Y_2$ of Y into two genus g handlebodies, each of which determines a product torus in the g -fold symmetric product of the Heegaard surface $\Sigma = \partial Y_1 = -\partial Y_2$. Deleting a marked point z from Σ to obtain an open surface, $\widehat{HF}(Y)$ is then defined as the Lagrangian Floer homology of the two tori T_1, T_2 in $\text{Sym}^g(\Sigma \setminus \{z\})$, see [9].

It is natural to ask how more general decompositions of 3-manifolds fit into this picture, and whether Heegaard-Floer theory can be viewed as a TQFT (at least in some partial sense). From the point of view of symplectic geometry, a natural answer is suggested by the work of Tim Perutz and Yankı Lekili. Namely, an elementary cobordism between two connected Riemann surfaces Σ_1, Σ_2 given by attaching a single handle determines a Lagrangian correspondence between appropriate symmetric products of Σ_1 and Σ_2 [10]. By composing these correspondences, one can associate to a 3-manifold with connected boundary Σ of genus g a *generalized Lagrangian submanifold* (cf. [16]) of the symmetric product $\text{Sym}^g(\Sigma \setminus \{z\})$. Recent work of Lekili and Perutz [4] shows that, given a decomposition $Y = Y_1 \cup_{\Sigma} Y_2$ of a closed 3-manifold, the (quilted) Floer homology of the generalized Lagrangian submanifolds of $\text{Sym}^g(\Sigma \setminus \{z\})$ determined by Y_1 and Y_2 recovers $\widehat{HF}(Y)$.

From a more combinatorial perspective, the *bordered Heegaard-Floer homology* of Robert Lipshitz, Peter Ozsváth and Dylan Thurston [6] associates to a parame-

terized Riemann surface F with connected boundary a finite dimensional differential algebra $\mathcal{A}(F)$ over \mathbb{Z}_2 , and to a 3-manifold Y with boundary $\partial Y = F \cup D^2$ a right A_∞ -module $\widehat{CFA}(Y)$ over $\mathcal{A}(F)$, as well as a left dg-module $\widehat{CFD}(Y)$ over $\mathcal{A}(-F)$. The main result of [6] shows that, given a decomposition of a closed 3-manifold $Y = Y_1 \cup Y_2$ with $\partial Y_1 = -\partial Y_2 = F \cup D^2$, $\widehat{HF}(Y)$ can be computed in terms of the A_∞ -tensor product of the modules associated to Y_1 and Y_2 , namely

$$\widehat{HF}(Y) \simeq H_*(\widehat{CFA}(Y_1) \otimes_{\mathcal{A}(F)} \widehat{CFD}(Y_2)).$$

In order to connect these two approaches, we consider a *partially wrapped* version of Floer theory for product Lagrangians in symmetric products of open Riemann surfaces. Concretely, given a Riemann surface with boundary F , a finite collection Z of marked points on ∂F , and an integer $k \geq 0$, we consider a partially wrapped Fukaya category $\mathcal{F}(\text{Sym}^k(F), Z)$, which differs from the usual (compactly supported) Fukaya category by the inclusion of additional objects, namely products of disjoint properly embedded arcs in F with boundary in $\partial F \setminus Z$. A nice feature of these categories is that they admit explicit sets of generating objects:

Theorem 1. *Let F be a compact Riemann surface with non-empty boundary, Z a finite subset of ∂F , and $\underline{\alpha} = \{\alpha_1, \dots, \alpha_n\}$ a collection of disjoint properly embedded arcs in F with boundary in $\partial F \setminus Z$. Assume that $F \setminus (\alpha_1 \cup \dots \cup \alpha_n)$ is a union of discs, each of which contains at most one point of Z . Then for $0 \leq k \leq n$, the partially wrapped Fukaya category $\mathcal{F}(\text{Sym}^k(F), Z)$ is generated by the $\binom{n}{k}$ Lagrangian submanifolds $D_s = \prod_{i \in s} \alpha_i$, where s ranges over all k -element subsets of $\{1, \dots, n\}$.*

To a decorated surface $\mathbb{F} = (F, Z, \underline{\alpha} = \{\alpha_i\})$ we can associate an A_∞ -algebra

$$\mathcal{A}(\mathbb{F}, k) = \bigoplus_{s,t} \text{hom}_{\mathcal{F}(\text{Sym}^k(F), Z)}(D_s, D_t). \quad (1)$$

The following special case is of particular interest:

Theorem 2. *Assume that F has a single boundary component, $|Z| = 1$, and the arcs $\alpha_1, \dots, \alpha_n$ ($n = 2g(F)$) decompose F into a single disc. Then $\mathcal{A}(\mathbb{F}, k)$ coincides with Lipshitz-Ozsváth-Thurston's bordered algebra [6].*

(The result remains true in greater generality, the only key requirement being that every component of $F \setminus (\alpha_1 \cup \dots \cup \alpha_n)$ should contain at least one point of Z .)

Now, consider a *sutured* 3-manifold Y , i.e. a 3-manifold Y with non-empty boundary, equipped with a decomposition $\partial Y = (-F_-) \cup F_+$, where F_\pm are oriented surfaces with boundary. Assume moreover that ∂Y and F_\pm are connected, and denote by g_\pm the genus of F_\pm . Given two integers k_\pm such that $k_+ - k_- = g_+ - g_-$ and a suitable Morse function on Y , Perutz's construction associates to Y a generalized Lagrangian correspondence (i.e. a formal composition of Lagrangian correspondences) \mathbb{T}_Y from $\text{Sym}^{k_-}(F_-)$ to $\text{Sym}^{k_+}(F_+)$. By the main result of [4] this correspondence is essentially independent of the chosen Morse function.

Picking a finite set of marked points $Z \subset \partial F_- = \partial F_+$, and two collections of disjoint arcs $\underline{\alpha}_-$ and $\underline{\alpha}_+$ on F_- and F_+ , we have two decorated surfaces $\mathbb{F}_\pm = (F_\pm, Z, \underline{\alpha}_\pm)$, and collections of product Lagrangian submanifolds $D_{\pm, s}$ ($s \in \mathcal{S}_\pm$) in $\text{Sym}^{k_\pm}(F_\pm)$ (namely, all products of k_\pm of the arcs in $\underline{\alpha}_\pm$). By a Yoneda-style construction, the correspondence \mathbb{T}_Y then determines an A_∞ -bimodule

$$\mathcal{Y}(\mathbb{T}_Y) = \bigoplus_{(s,t) \in \mathcal{S}_- \times \mathcal{S}_+} \text{hom}(D_{-,s}, \mathbb{T}_Y, D_{+,t}) \in \mathcal{A}(\mathbb{F}_-, k_-)\text{-mod-}\mathcal{A}(\mathbb{F}_+, k_+), \quad (2)$$

where $\text{hom}(D_{-,s}, \mathbb{T}_Y, D_{+,t})$ is defined in terms of quilted Floer complexes [8, 16, 17] after suitably perturbing $D_{-,s}$ and $D_{+,t}$ by partial wrapping along the boundary. A slightly different but equivalent definition is as follows. With quite a bit of extra work, via the Ma'u-Wehrheim-Woodward machinery the correspondence \mathbb{T}_Y defines an A_∞ -functor Φ_Y from $\mathcal{F}(\text{Sym}^{k_-}(F_-), Z)$ to a suitable enlargement of $\mathcal{F}(\text{Sym}^{k_+}(F_+), Z)$; with this understood, $\mathcal{Y}(\mathbb{T}_Y) \simeq \bigoplus_{(s,t)} \text{hom}(\Phi_Y(D_{-,s}), D_{+,t})$.

The A_∞ -bimodules $\mathcal{Y}(\mathbb{T}_Y)$ are expected to obey the following gluing property:

Conjecture 3. *Let F, F', F'' be connected Riemann surfaces and Z a finite subset of $\partial F \simeq \partial F' \simeq \partial F''$. Let Y_1, Y_2 be two sutured manifolds with $\partial Y_1 = (-F) \cup F'$ and $\partial Y_2 = (-F') \cup F''$, and let $Y = Y_1 \cup_{F'} Y_2$ be the sutured manifold obtained by gluing Y_1 and Y_2 along F' . Equip F, F', F'' with collections of disjoint properly embedded arcs $\underline{\alpha}, \underline{\alpha}', \underline{\alpha}''$, and assume that $\underline{\alpha}'$ decomposes F' into a union of discs each containing at most one point of Z . Then*

$$\mathcal{Y}(\mathbb{T}_Y) \simeq \mathcal{Y}(\mathbb{T}_{Y_1}) \otimes_{\mathcal{A}(\mathbb{F}', k')} \mathcal{Y}(\mathbb{T}_{Y_2}). \quad (3)$$

In its most general form this statement relies on results in Floer theory for generalized Lagrangian correspondences which are not yet fully established, hence we state it as a conjecture; however, we believe that a proof should be within reach of standard techniques.

As a special case, let F be a genus g surface with connected boundary, decorated with a single point $z \in \partial F$ and a collection of $2g$ arcs cutting F into a disc. Then to a 3-manifold Y_1 with boundary $\partial Y_1 = F \cup D^2$ we can associate a generalized Lagrangian submanifold \mathbb{T}_{Y_1} of $\text{Sym}^g(F)$, and an A_∞ -module $\mathcal{Y}(\mathbb{T}_{Y_1}) = \bigoplus_s \text{hom}(\mathbb{T}_{Y_1}, D_s) \in \text{mod-}\mathcal{A}(\mathbb{F}, g)$. Viewing \mathbb{T}_{Y_1} as a generalized correspondence from $\text{Sym}^g(-F)$ to $\text{Sym}^0(D^2) = \{\text{pt}\}$ instead, we obtain a left A_∞ -module over $\mathcal{A}(-\mathbb{F}, g)$. However, $\mathcal{A}(-\mathbb{F}, g) = \mathcal{A}(\mathbb{F}, g)^{op}$, and the two constructions yield the same module. If now we have another 3-manifold Y_2 with $\partial Y_2 = -F \cup D^2$, we can associate to it a generalized Lagrangian submanifold \mathbb{T}_{Y_2} in $\text{Sym}^g(-F)$ or, after orientation reversal, \mathbb{T}_{-Y_2} in $\text{Sym}^g(F)$. This yields A_∞ -modules $\mathcal{Y}(\mathbb{T}_{Y_2}) \in \text{mod-}\mathcal{A}(-\mathbb{F}, g) \simeq \mathcal{A}(\mathbb{F}, g)\text{-mod}$, and $\mathcal{Y}(\mathbb{T}_{-Y_2}) \in \text{mod-}\mathcal{A}(\mathbb{F}, g)$.

Theorem 4. *With this understood, and denoting by Y the closed 3-manifold obtained by gluing Y_1 and Y_2 along their boundaries, we have quasi-isomorphisms*

$$\begin{aligned} \widehat{\mathcal{C}F}(Y) &\simeq \text{hom}_{\mathcal{F}^\#(\text{Sym}^g(F))}(\mathbb{T}_{Y_1}, \mathbb{T}_{-Y_2}) \simeq \text{hom}_{\text{mod-}\mathcal{A}(\mathbb{F}, g)}(\mathcal{Y}(\mathbb{T}_{-Y_2}), \mathcal{Y}(\mathbb{T}_{Y_1})) \\ &\simeq \mathcal{Y}(\mathbb{T}_{Y_1}) \otimes_{\mathcal{A}(\mathbb{F}, g)} \mathcal{Y}(\mathbb{T}_{Y_2}). \end{aligned} \quad (4)$$

In fact, $\mathcal{Y}(\mathbb{T}_{Y_i})$ is quasi-isomorphic to the bordered A_∞ -module $\widehat{CFA}(Y_i)$. In light of this, it is instructive to compare Theorem 4 with the pairing theorem obtained by Lipshitz, Ozsváth and Thurston [6]: even though $\widehat{CFA}(Y_i)$ and $\widehat{CFD}(Y_i)$ seem very different at first glance (and even at second glance), our result suggests that they can in fact be used interchangeably.

The rest of this paper is structured as follows: first, in section 2 we explain how Heegaard-Floer homology can be understood in terms of Lagrangian correspondences, following the work of Perutz and Lekili [10, 4]. Then in section 3 we introduce partially wrapped Fukaya categories of symmetric products, and sketch the proofs of Theorems 1 and 2. In section 4 we briefly discuss Yoneda embedding as well as Conjecture 3 and Theorem 4. Finally, in section 5 we discuss the relation with bordered Heegaard-Floer homology.

The reader will not find detailed proofs for any of the statements here, nor a general discussion of partially wrapped Fukaya categories. Some of the material is treated in greater depth in the preprint [2], the rest will appear in a future paper.

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2. Heegaard-Floer homology from Lagrangian correspondences

2.1. Lagrangian correspondences. A *Lagrangian correspondence* between two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is, by definition, a Lagrangian submanifold of the product $M_1 \times M_2$ equipped with the product symplectic form $(-\omega_1) \oplus \omega_2$. Lagrangian correspondences can be thought of as a far-reaching generalization of symplectomorphisms (whose graphs are examples of correspondences); in particular, under suitable transversality assumptions we can consider the *composition* of two correspondences $L_{01} \subset M_0 \times M_1$ and $L_{12} \subset M_1 \times M_2$,

$$L_{01} \circ L_{12} = \{(x, z) \in M_0 \times M_2 \mid \exists y \in M_1 \text{ s.t. } (x, y) \in L_{01} \text{ and } (y, z) \in L_{12}\}.$$

The image of a Lagrangian submanifold $L_1 \subset M_1$ by a Lagrangian correspondence $L_{12} \subset M_1 \times M_2$ is defined similarly, viewing L_1 as a correspondence from $\{pt\}$ to M_1 . Unfortunately, in general the geometric composition is not a smooth embedded Lagrangian. Nonetheless, we can enlarge symplectic geometry by considering *generalized Lagrangian correspondences*, i.e. sequences of Lagrangian correspondences (interpreted as formal compositions), and *generalized Lagrangian submanifolds*, i.e. generalized Lagrangian correspondences from $\{pt\}$ to a given symplectic manifold.

The work of Ma'u, Wehrheim and Woodward (see e.g. [16, 17, 8]) shows that Lagrangian Floer theory behaves well with respect to (generalized) correspondences. Given a sequence of Lagrangian correspondences $L_{i-1,i} \subset M_{i-1} \times M_i$ ($i = 1, \dots, n$), with $M_0 = M_n = \{pt\}$, the *quilted Floer complex* $CF(L_{0,1}, \dots, L_{n-1,n})$ is generated by *generalized intersections*, i.e. tuples $(x_1, \dots, x_{n-1}) \in M_1 \times \dots \times M_{n-1}$ such that $(x_{i-1}, x_i) \in L_{i-1,i}$ for all i , and carries a differential which counts “quilted pseudoholomorphic strips” in $M_1 \times \dots \times M_{n-1}$. Under suitable technical assumptions (e.g., monotonicity), Lagrangian Floer theory carries over to this setting.

Thus, Ma'u, Wehrheim and Woodward associate to a monotone symplectic manifold (M, ω) its *extended Fukaya category* $\mathcal{F}^\#(M)$, whose objects are monotone generalized Lagrangian submanifolds and in which morphisms are given by quilted Floer complexes. Composition of morphisms is defined by counting quilted pseudoholomorphic discs, and as in usual Floer theory, it is only associative up to homotopy, so $\mathcal{F}^\#(M)$ is an A_∞ -category. The key property of these extended Fukaya categories is that a monotone (generalized) Lagrangian correspondence L_{12} from M_1 to M_2 induces an A_∞ -functor from $\mathcal{F}^\#(M_1)$ to $\mathcal{F}^\#(M_2)$, which on the level of objects is simply concatenation with L_{12} . Moreover, composition of Lagrangian correspondences matches with composition of A_∞ -functors [8].

Remark. By construction, the usual Fukaya category $\mathcal{F}(M)$ admits a fully faithful embedding as a subcategory of $\mathcal{F}^\#(M)$. In fact, $\mathcal{F}^\#(M)$ embeds into the category of A_∞ -modules over the usual Fukaya category, so although generalized Lagrangian correspondences play an important conceptual role in our discussion, they only enlarge the Fukaya category in a fairly mild manner.

2.2. Symmetric products. As mentioned in the introduction, work in progress of Lekili and Perutz [4] shows that Heegaard-Floer homology can be understood in terms of quilted Floer homology for Lagrangian correspondences between symmetric products. The relevant correspondences were introduced by Perutz in his thesis [10].

Let Σ be an open Riemann surface (with infinite cylindrical ends, i.e., the complement of a finite set in a compact Riemann surface), equipped with an area form σ . We consider the symmetric product $\text{Sym}^k(\Sigma)$, equipped with the product complex structure J , and a Kähler form ω which coincides with the product Kähler form on Σ^k away from the diagonal strata. Following Perutz we choose ω so that its cohomology class is negatively proportional to $c_1(T\text{Sym}^k(\Sigma))$.

Let γ be a non-separating simple closed curve on Σ , and Σ_γ the surface obtained from Σ by deleting a tubular neighborhood of γ and gluing in two discs. Equip Σ_γ with a complex structure which agrees with that of Σ away from γ , and equip $\text{Sym}^k(\Sigma)$ and $\text{Sym}^{k-1}(\Sigma_\gamma)$ with Kähler forms ω and ω_γ chosen as above.

Theorem 5 (Perutz [10]). *The simple closed curve γ determines a Lagrangian correspondence T_γ in the product $(\text{Sym}^{k-1}(\Sigma_\gamma) \times \text{Sym}^k(\Sigma), -\omega_\gamma \oplus \omega)$, canonically up to Hamiltonian isotopy.*

Given r disjoint simple closed curves $\gamma_1, \dots, \gamma_r$, linearly independent in $H_1(\Sigma)$, we can consider the sequence of correspondences that arise from successive surgeries

along $\gamma_1, \dots, \gamma_r$. The main properties of these correspondences (see Theorem A in [10]) imply immediately that their composition defines an embedded Lagrangian correspondence $T_{\gamma_1, \dots, \gamma_r}$ in $\text{Sym}^{k-r}(\Sigma_{\gamma_1, \dots, \gamma_r}) \times \text{Sym}^k(\Sigma)$.

When $r = k = g(\Sigma)$, this construction yields a Lagrangian torus in $\text{Sym}^k(\Sigma)$, which by [10, Lemma 3.20] is smoothly isotopic to the product torus $\gamma_1 \times \dots \times \gamma_k$; Lekili and Perutz show that these two tori are in fact Hamiltonian isotopic [4].

Remark. We are not quite in the setting considered by Ma'u, Wehrheim and Woodward, but Floer theory remains well behaved thanks to two key properties of the Lagrangian submanifolds under consideration: their relative π_2 is trivial (which prevents bubbling), and they are *balanced*. (A Lagrangian submanifold in a monotone symplectic manifold is said to be balanced if the holonomy of a fixed connection 1-form with curvature equal to the symplectic form vanishes on it; this is a natural analogue of the notion of exact Lagrangian submanifold in an exact symplectic manifold). The balancing condition is closely related to admissibility of Heegaard diagrams, and ensures that the symplectic area of a pseudo-holomorphic strip connecting two given intersection points is determined *a priori* by its Maslov index (cf. [17, Lemma 4.1.5]). This property is what allows us to work over \mathbb{Z}_2 rather than over a Novikov field.

2.3. Heegaard-Floer homology. Consider a closed 3-manifold Y , and a Morse function $f : Y \rightarrow \mathbb{R}$ (with only one minimum and one maximum, and with distinct critical values). Then the complement Y' of a ball in Y (obtained by deleting a neighborhood of a Morse trajectory from the maximum to the minimum) can be decomposed into a succession of elementary cobordisms Y'_i ($i = 1, \dots, r$) between connected Riemann surfaces with boundary $\Sigma_0, \Sigma_1, \dots, \Sigma_r$ (where $\Sigma_0 = \Sigma_r = D^2$, and the genus increases or decreases by 1 at each step). By Theorem 5, each Y'_i determines a Lagrangian correspondence $L_i \subset \text{Sym}^{g_{i-1}}(\Sigma_{i-1}) \times \text{Sym}^{g_i}(\Sigma_i)$ between the relevant symmetric products (here g_i is the genus of Σ_i , and we implicitly complete Σ_i by attaching to it an infinite cylindrical end). By the work of Lekili and Perutz [4], the quilted Floer homology of the sequence (L_1, \dots, L_r) is independent of the choice of the Morse function f and isomorphic to $\widehat{HF}(Y)$.

More generally, consider a sutured 3-manifold Y , i.e. a 3-manifold whose boundary is decomposed into a union $(-F_-) \cup_{\Gamma} F_+$, where F_{\pm} are connected oriented surfaces of genus g_{\pm} with boundary $\partial F_- \simeq \partial F_+ \simeq \Gamma$. Shrinking F_{\pm} slightly within ∂Y , it is advantageous to think of the boundary of Y as consisting actually of *three* pieces, $\partial Y = (-F_-) \cup (\Gamma \times [0, 1]) \cup F_+$. By considering a Morse function $f : Y \rightarrow [0, 1]$ with index 1 and 2 critical points only, with $f^{-1}(1) = F_-$ and $f^{-1}(0) = F_+$, we can view Y as a succession of elementary cobordisms between connected Riemann surfaces with boundary, starting with F_- and ending with F_+ . As above, Perutz's construction associates a Lagrangian correspondence to each of these elementary cobordisms. Thus we can associate to Y a generalized Lagrangian correspondence $\mathbb{T}_Y = \mathbb{T}_{Y, k_{\pm}}$ from $\text{Sym}^{k_-}(F_-)$ to $\text{Sym}^{k_+}(F_+)$ whenever $k_+ - k_- = g_+ - g_-$. The generalized correspondence \mathbb{T}_Y can be viewed either as an object of the extended Fukaya category $\mathcal{F}^{\#}(\text{Sym}^{k_-}(-F_-) \times \text{Sym}^{k_+}(F_+))$, or as an A_{∞} -functor from $\mathcal{F}^{\#}(\text{Sym}^{k_-}(F_-))$ to $\mathcal{F}^{\#}(\text{Sym}^{k_+}(F_+))$.

Theorem 6 (Lekili-Perutz [4]). *Up to quasi-isomorphism the object \mathbb{T}_Y is independent of the choice of Morse function on Y .*

Given two sutured manifolds Y_1 and Y_2 ($\partial Y_i = (-F_{i,-}) \cup F_{i,+}$) and a diffeomorphism $\phi : F_{1,+} \rightarrow F_{2,-}$, gluing Y_1 and Y_2 by identifying the positive boundary of Y_1 with the negative boundary of Y_2 via ϕ yields a new sutured manifold Y' . As a cobordism from $F_{1,-}$ to $F_{2,+}$, Y' is simply the concatenation of the cobordisms Y_1 and Y_2 . Hence, the generalized Lagrangian correspondence $\mathbb{T}_{Y'}$ associated to Y' is just the (formal) composition of \mathbb{T}_{Y_1} and \mathbb{T}_{Y_2} .

The case where Y_1 is a cobordism from the disc D^2 to a genus g surface F (with a single boundary component) and Y_2 is a cobordism from F to D^2 (so $\partial Y_1 \simeq -\partial Y_2 \simeq F \cup_{S^1} D^2$) is of particular interest. In that case, we associate to Y_1 a generalized correspondence from $\text{Sym}^0(D^2) = \{pt\}$ to $\text{Sym}^g(F)$, i.e. an object \mathbb{T}_{Y_1} of $\mathcal{F}^\#(\text{Sym}^g(F))$, and to Y_2 a generalized correspondence \mathbb{T}_{Y_2} from $\text{Sym}^g(F)$ to $\text{Sym}^0(D^2) = \{pt\}$, i.e. a generalized Lagrangian submanifold of $\text{Sym}^g(-F)$. Reversing the orientation of Y_2 , i.e. viewing $-Y_2$ as the opposite cobordism from D^2 to F , we get a generalized Lagrangian submanifold \mathbb{T}_{-Y_2} in $\text{Sym}^g(F)$, which differs from \mathbb{T}_{Y_2} simply by orientation reversal. Denoting by $Y (= Y' \cup B^3)$ the closed 3-manifold obtained by gluing Y_1 and Y_2 along their entire boundary, the result of [4] now says that

$$\widehat{CF}(Y) \simeq CF(\mathbb{T}_{Y_1}, \mathbb{T}_{Y_2}) \simeq \text{hom}_{\mathcal{F}^\#(\text{Sym}^g(F))}(\mathbb{T}_{Y_1}, \mathbb{T}_{-Y_2}). \quad (5)$$

3. Partially wrapped Fukaya categories of symmetric products

3.1. Positive perturbations and partial wrapping. Let F be a connected Riemann surface with non-empty boundary, and Z a finite subset of ∂F . Assume for now that every connected component of ∂F contains at least one point of Z . Then the components of $\partial F \setminus Z$ are open intervals, and carry a natural orientation induced by that of F .

Definition 7. *Let $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$, $\underline{\lambda}' = (\lambda'_1, \dots, \lambda'_k)$ be two k -tuples of disjoint properly embedded arcs in F , with boundary in $\partial F \setminus Z$. We say that the pair $(\underline{\lambda}, \underline{\lambda}')$ is positive, and write $\underline{\lambda} > \underline{\lambda}'$, if along each component of $\partial F \setminus Z$ the points of $\partial(\bigcup_i \lambda'_i)$ all lie before those of $\partial(\bigcup_i \lambda_i)$.*

Similarly, given tuples $\underline{\lambda}^j = (\lambda_1^j, \dots, \lambda_k^j)$ ($j = 0, \dots, \ell$), we say that the sequence $(\underline{\lambda}^0, \dots, \underline{\lambda}^\ell)$ is positive if each pair $(\underline{\lambda}^j, \underline{\lambda}^{j+1})$ is positive, i.e. $\underline{\lambda}^0 > \dots > \underline{\lambda}^\ell$.

Given two tuples $\underline{\lambda} = (\lambda_1, \dots, \lambda_k)$ and $\underline{\lambda}' = (\lambda'_1, \dots, \lambda'_k)$, we can perturb each arc λ_i (resp. λ'_i) by an isotopy that pushes it in the positive (resp. negative) direction along ∂F , without crossing Z , to obtain new tuples $\tilde{\underline{\lambda}} = (\tilde{\lambda}_1, \dots, \tilde{\lambda}_k)$ and $\tilde{\underline{\lambda}}' = (\tilde{\lambda}'_1, \dots, \tilde{\lambda}'_k)$ with the property that $\tilde{\underline{\lambda}} > \tilde{\underline{\lambda}}'$. Similarly, any sequence $(\underline{\lambda}^0, \dots, \underline{\lambda}^\ell)$ can be made into a positive sequence by means of suitable isotopies supported near ∂F (again, the isotopies are not allowed to cross Z).

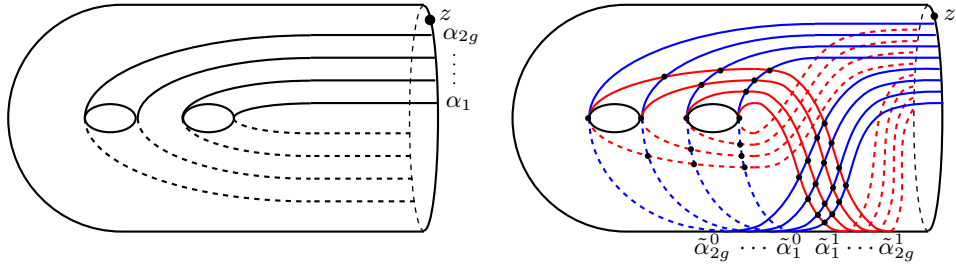


Figure 1. The arcs α_i and $\tilde{\alpha}_i^j$ on $(F, \{z\})$

Example. Let $\underline{\alpha} = (\alpha_1, \dots, \alpha_{2g})$ be the tuple of arcs represented on Figure 1 left: then the perturbed tuples $\tilde{\underline{\alpha}}^j = (\tilde{\alpha}_1^j, \dots, \tilde{\alpha}_{2g}^j)$ (Figure 1 right) satisfy $\tilde{\underline{\alpha}}^0 > \tilde{\underline{\alpha}}^1$, i.e. the pair $(\tilde{\underline{\alpha}}^0, \tilde{\underline{\alpha}}^1)$ is a positive perturbation of $(\underline{\alpha}, \underline{\alpha})$.

Next, consider a sequence (L_0, \dots, L_ℓ) of Lagrangian submanifolds in the symmetric product $\text{Sym}^k(F)$, each of which is either a closed submanifold contained in the interior of $\text{Sym}^k(F)$ or a product of disjoint properly embedded arcs $L_j = \lambda_1^j \times \dots \times \lambda_k^j$. Then we say that the sequence (L_0, \dots, L_ℓ) is positive if, whenever L_i and L_j are products of disjointly embedded arcs for $i < j$, the corresponding k -tuples of arcs satisfy $\underline{\lambda}^i > \underline{\lambda}^j$. (There is no condition on the closed Lagrangians). Modifying the arcs $\lambda_1^j, \dots, \lambda_k^j$ by suitable isotopies supported near ∂F (without crossing Z) as above, given any sequence (L_0, \dots, L_ℓ) we can construct Lagrangian submanifolds $\tilde{L}_0, \dots, \tilde{L}_\ell$ such that: (1) \tilde{L}_i is Hamiltonian isotopic to L_i , and either contained in the interior of $\text{Sym}^k(F)$ or a product of disjoint properly embedded arcs; and (2) the sequence $(\tilde{L}_0, \dots, \tilde{L}_\ell)$ is positive. We call $(\tilde{L}_0, \dots, \tilde{L}_\ell)$ a *positive perturbation* of the sequence (L_0, \dots, L_ℓ) .

With this understood, we can now give an informal (and imprecise) definition of the partially wrapped Fukaya category of the symmetric product $\text{Sym}^k(F)$ relative to the set Z ; we are still assuming that every component of ∂F contains at least one point of Z . The reader is referred to [2] for a more precise construction.

Definition 8. *The partially wrapped Fukaya category $\mathcal{F} = \mathcal{F}(\text{Sym}^k(F), Z)$ is an A_∞ -category with objects of two types:*

1. *closed balanced Lagrangian submanifolds lying in the interior of $\text{Sym}^k(F)$;*
2. *properly embedded Lagrangian submanifolds of the form $\lambda_1 \times \dots \times \lambda_k$, where λ_i are disjoint properly embedded arcs with boundary contained in $\partial F \setminus Z$.*

Morphism spaces and compositions are defined by perturbing objects of the second type in a suitable manner near the boundary so that they form positive sequences. Namely, we set $\text{hom}_{\mathcal{F}}(L_0, L_1) = CF(\tilde{L}_0, \tilde{L}_1)$ (i.e., the \mathbb{Z}_2 -vector space generated by points of $\tilde{L}_0 \cap \tilde{L}_1$, with a differential counting rigid holomorphic discs), where $(\tilde{L}_0, \tilde{L}_1)$ is a suitably chosen positive perturbation of the pair (L_0, L_1) . The composition $m_2 : \text{hom}_{\mathcal{F}}(L_0, L_1) \otimes \text{hom}_{\mathcal{F}}(L_1, L_2) \rightarrow \text{hom}_{\mathcal{F}}(L_0, L_2)$ and higher products $m_\ell : \text{hom}_{\mathcal{F}}(L_0, L_1) \otimes \dots \otimes \text{hom}_{\mathcal{F}}(L_{\ell-1}, L_\ell) \rightarrow \text{hom}_{\mathcal{F}}(L_0, L_\ell)$ are similarly defined

by perturbing (L_0, \dots, L_ℓ) to a positive sequence $(\tilde{L}_0, \dots, \tilde{L}_\ell)$ and counting rigid holomorphic discs with boundary on the perturbed Lagrangians.

The extended category $\mathcal{F}^\# = \mathcal{F}^\#(\mathrm{Sym}^k(F), Z)$ is defined similarly, but also includes closed balanced generalized Lagrangian submanifolds of $\mathrm{Sym}^k(F)$ (i.e., formal images of Lagrangians under sequences of balanced Lagrangian correspondences) of the sort introduced in §2.

To be more precise, the construction of the partially wrapped Fukaya category involves the completion $\hat{F} = F \cup (\partial F \times [1, \infty))$, and its symmetric product $\mathrm{Sym}^k(\hat{F})$. Arcs in F can be completed to properly embedded arcs in \hat{F} , translation-invariant in the cylindrical ends, and hence the objects of $\mathcal{F}(\mathrm{Sym}^k(F), Z)$ can be completed to properly embedded Lagrangian submanifolds of $\mathrm{Sym}^k(\hat{F})$ which are cylindrical at infinity. The Riemann surface \hat{F} carries a Hamiltonian vector field supported away from the interior of F and whose positive (resp. negative) time flow rotates the cylindrical ends of \hat{F} in the positive (resp. negative) direction and accumulates towards the rays $Z \times [1, \infty)$. (In the cylindrical ends $\partial F \times [1, \infty)$, the generating Hamiltonian function h is of the form $h(x, r) = \rho(x)r$ where $\rho : \partial F \rightarrow [0, 1]$ satisfies $\rho^{-1}(0) = Z$). This flow on \hat{F} can be used to construct a Hamiltonian flow on $\mathrm{Sym}^k(\hat{F})$ which preserves the product structure away from the diagonal (namely, the generating Hamiltonian is given by $H(\{z_1, \dots, z_k\}) = \sum_i h(z_i)$ away from the diagonal). The A_∞ -category $\mathcal{F}(\mathrm{Sym}^k(F), Z)$ is then constructed in essentially the same manner as the wrapped Fukaya category defined by Abouzaid and Seidel [1]: namely, morphism spaces are limits of the Floer complexes upon long-time perturbation by the Hamiltonian flow. (In general various technical issues could arise with this construction, but product Lagrangians in $\mathrm{Sym}^k(\hat{F})$ are fairly well-behaved, see [2]).

When a component of ∂F does not contain any point of Z , the Hamiltonian flow that we consider rotates the corresponding cylindrical end of \hat{F} by arbitrarily large amounts. Hence the perturbation causes properly embedded arcs in \hat{F} to wrap around the cylindrical end infinitely many times, which typically makes the complex $\mathrm{hom}_{\mathcal{F}}(L_0, L_1)$ infinitely generated when L_0 and L_1 are non-compact objects of $\mathcal{F}(\mathrm{Sym}^k(F), Z)$. For instance, when $Z = \emptyset$ the category we consider is simply the wrapped Fukaya category of $\mathrm{Sym}^k(\hat{F})$ as defined in [1].

3.2. The algebra of a decorated surface.

Definition 9. A decorated surface is a triple $\mathbb{F} = (F, Z, \underline{\alpha})$ where F is a connected compact Riemann surface with non-empty boundary, Z is a finite subset of ∂F , and $\underline{\alpha} = \{\alpha_1, \dots, \alpha_n\}$ is a collection of disjoint properly embedded arcs in F with boundary in $\partial F \setminus Z$.

Given a decorated surface $\mathbb{F} = (F, Z, \underline{\alpha})$, an integer $k \leq n$, and a k -element subset $s \subseteq \{1, \dots, n\}$, the product $D_s = \prod_{i \in s} \alpha_i$ is an object of $\mathcal{F} = \mathcal{F}(\mathrm{Sym}^k(F), Z)$. The endomorphism algebra of the direct sum of these objects is an A_∞ -algebra naturally associated to \mathbb{F} .

Definition 10. For $k \leq n$, denote by \mathcal{S}_k^n the set of all k -element subsets of $\{1, \dots, n\}$. Then to a decorated surface $\mathbb{F} = (F, Z, \underline{\alpha} = \{\alpha_1, \dots, \alpha_n\})$ and an integer $k \leq n$ we associate the A_∞ -algebra

$$\mathcal{A}(\mathbb{F}, k) = \bigoplus_{s, t \in \mathcal{S}_k^n} \text{hom}_{\mathcal{F}}(D_s, D_t), \quad \text{where } D_s = \prod_{i \in s} \alpha_i,$$

with differential and products defined by those of $\mathcal{F} = \mathcal{F}(\text{Sym}^k(F), Z)$.

In the rest of this section, we focus on a special case where $\mathcal{A}(\mathbb{F}, k)$ can be expressed in purely combinatorial terms, and is in fact isomorphic to (the obvious generalization of) the *bordered algebra* introduced by Lipshitz, Ozsváth and Thurston [6]. The following proposition implies Theorem 2 as a special case:

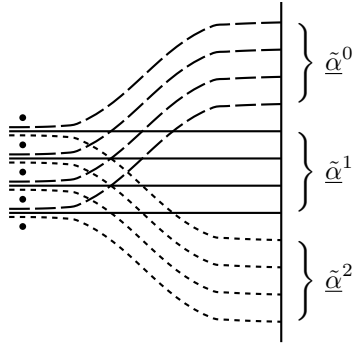
Proposition 11. Let $\mathbb{F} = (F, Z, \underline{\alpha})$ be a decorated surface, and assume that every connected component of $F \setminus (\alpha_1 \cup \dots \cup \alpha_n)$ contains at least one point of Z . For $i, j \in \{1, \dots, n\}$, denote by χ_i^j the set of chords from $\partial\alpha_i$ to $\partial\alpha_j$ in $\partial F \setminus Z$, i.e. homotopy classes of immersed arcs $\gamma : [0, 1] \rightarrow \partial F \setminus Z$ such that $\gamma(0) \in \partial\alpha_i$, $\gamma(1) \in \partial\alpha_j$, and the tangent vector $\gamma'(t)$ is always oriented in the positive direction along ∂F . Moreover, denote $\bar{\chi}_i^i$ the set obtained by adjoining to χ_i^i an extra element $\mathbf{1}_i$, and let $\bar{\chi}_i^j = \chi_i^j$ for $i \neq j$. Then the following properties hold:

- Given $s, t \in \mathcal{S}_k^n$, let $s = \{i_1, \dots, i_k\}$, and denote by $\Phi(s, t)$ the set of bijective maps from s to t . Then the \mathbb{Z}_2 -vector space $\text{hom}_{\mathcal{F}(\text{Sym}^k(F), Z)}(D_s, D_t)$ admits a basis indexed by the elements of

$$\bar{\chi}_s^t := \bigsqcup_{f \in \Phi(s, t)} \left(\bar{\chi}_{i_1}^{f(i_1)} \times \dots \times \bar{\chi}_{i_k}^{f(i_k)} \right).$$

- The differential and product in $\mathcal{A}(\mathbb{F}, k)$ are determined by explicit combinatorial formulas as in [6].
- The higher products $\{m_\ell\}_{\ell \geq 3}$ vanish identically, i.e. the A_∞ -algebra $\mathcal{A}(\mathbb{F}, k)$ is in fact a differential algebra.

Sketch of proof (see also [2]). For $\ell \geq 1$, we construct perturbations $\tilde{\alpha}^0, \dots, \tilde{\alpha}^\ell$ of $\underline{\alpha}$, with $\tilde{\alpha}^0 > \dots > \tilde{\alpha}^\ell$, in such a way that the diagram formed by the $\ell + 1$ collections of n arcs $\tilde{\alpha}_i^j$ on F enjoys properties similar to those of “nice” diagrams in Heegaard-Floer theory (cf. [13]). Namely, we ask that for each i the arcs $\tilde{\alpha}_i^0, \dots, \tilde{\alpha}_i^\ell$ remain close to α_i in the interior of F , where any two of them intersect transversely exactly once; the total number of intersections in the diagram is minimal; and all intersections between the arcs of $\tilde{\alpha}^j$ and those of $\tilde{\alpha}^{j'}$ are transverse and occur with the same oriented angle $(j - j')\theta$ (for a fixed small $\theta > 0$) between the two arcs at the intersection point. Hence the local picture near any interval component of $\partial F \setminus Z$ is as shown in Figure 2. (At a component of ∂F which does not carry a point of Z , we need to consider arcs which wrap infinitely many times around the cylindrical end of the completed surface \hat{F} , but the situation is otherwise unchanged).

Figure 2. Positive perturbations of $\underline{\alpha}$ near ∂F

For $j < j'$ and $i, i' \in \{1, \dots, n\}$ we have a natural bijection between the points of $\tilde{\alpha}_i^j \cap \tilde{\alpha}_{i'}^{j'}$ and the elements of $\tilde{\chi}_i^{j'}$. Hence, passing to the symmetric product, the intersections of $\tilde{D}_s^j = \prod_{i \in s} \tilde{\alpha}_i^j$ and $\tilde{D}_t^{j'} = \prod_{i' \in t} \tilde{\alpha}_{i'}^{j'}$ are transverse and in bijection with the elements of $\tilde{\chi}_s^t$. The first claim follows.

The rest of the proposition follows from a calculation of the Maslov index of a holomorphic disc in $\text{Sym}^k(F)$ with boundary on $\ell + 1$ product Lagrangians $\tilde{D}_{s_0}^0, \dots, \tilde{D}_{s_\ell}^\ell$. Namely, let ϕ be the homotopy class of such a holomorphic disc contributing to the order ℓ product in $\mathcal{A}(\mathbb{F}, k)$. Projecting from the symmetric product to F , we can think of ϕ as a 2-chain in F with boundary on the arcs of the diagram, staying within the bounded regions of the diagram (i.e., those which do not intersect ∂F). Then the Maslov index $\mu(\phi)$ and the intersection number $i(\phi)$ of ϕ with the diagonal divisor in $\text{Sym}^k(F)$ are related to each other by the following formula due to Sarkar [12]:

$$\mu(\phi) = i(\phi) + 2e(\phi) - (\ell - 1)k/2, \quad (6)$$

where $e(\phi)$ is the *Euler measure* of the 2-chain ϕ , characterized by additivity and by the property that the Euler measure of an embedded m -gon with convex corners is $1 - \frac{m}{4}$. On the other hand, since every component of $F \setminus (\alpha_1 \cup \dots \cup \alpha_n)$ contains a point of Z , the regions of the diagram corresponding to those components remain unbounded after perturbation. In particular, the regions marked by dots in Figure 2 are all unbounded, and hence not part of the support of ϕ .

This implies that the support of ϕ is contained in a union of regions which are either planar (as in Figure 2) or cylindrical (in the case of a component of ∂F which does not carry any point of Z), and within which the Euler measure of a convex polygonal region can be computed by summing contributions from its vertices, namely $\frac{1}{4} - \frac{\vartheta}{2\pi}$ for a vertex with angle ϑ . Considering the respective contributions of the $(\ell + 1)k$ corners of the chain ϕ (and observing that the contributions from any other vertices traversed by the boundary of ϕ cancel out), we conclude that $e(\phi) = (\ell - 1)k/4$, and $\mu(\phi) = i(\phi) \geq 0$.

On the other hand, m_ℓ counts rigid holomorphic discs, for which $\mu(\phi) = 2 - \ell$. This immediately implies that $m_\ell = 0$ for $\ell \geq 3$. For $\ell = 1$, the diagram we

consider is “nice”, i.e. all the bounded regions are quadrilaterals; as observed by Sarkar and Wang, this implies that the Floer differential on $CF(\tilde{D}_s^0, \tilde{D}_t^1)$ counts empty embedded rectangles [13, Theorems 3.3 and 3.4]. Finally, for $\ell = 2$, the Maslov index formula shows that the product counts discs which are disjoint from the diagonal strata in $\text{Sym}^k(F)$. By an argument similar to that in [5] (see also [2, Proposition 3.5]), this implies that m_2 counts k -tuples of immersed holomorphic triangles in F which either are disjoint or overlap head-to-tail (cf. [5, Lemma 2.6]).

Finally, these combinatorial descriptions of m_1 and m_2 in terms of diagrams on F can be recast in terms of Lipshitz, Ozsváth and Thurston’s definition of differentials and products in the bordered algebra [6]. Namely, the dictionary between points of $\tilde{D}_s^0 \cap \tilde{D}_t^1$ proceeds by matching intersections of $\tilde{\alpha}_i^0$ with $\tilde{\alpha}_j^1$ near ∂F with chords from α_i to α_j (pictured as upwards strands in the notation of [6]), and the intersection of $\tilde{\alpha}_i^0$ with $\tilde{\alpha}_i^1$ in the interior of F with a pair of horizontal dotted lines in the graphical notation of [6]. See [2, section 3] for details. \square

3.3. Generating the partially wrapped Fukaya category. In this section, we outline the proof of Theorem 1. The main ingredients are Lefschetz fibrations on the symmetric product, their Fukaya categories as defined and studied by Seidel [14, 15], and acceleration A_∞ -functors between partially wrapped Fukaya categories.

3.3.1. Lefschetz fibrations on the symmetric product. Let \hat{F} be an open Riemann surface (with infinite cylindrical ends), and let $\pi : \hat{F} \rightarrow \mathbb{C}$ be a branched covering map. Assume that the critical points q_1, \dots, q_n of π are non-degenerate (i.e., the covering π is *simple*), and that the critical values $p_1, \dots, p_n \in \mathbb{C}$ are distinct, lie in the unit disc, and satisfy $\text{Im}(p_1) < \dots < \text{Im}(p_n)$.

Each critical point q_j of π determines a properly embedded arc $\hat{\alpha}_j \subset \hat{F}$, namely the union of the two lifts of the half-line $\mathbb{R}_{\geq 0} + p_j$ which pass through q_j .

We consider the k -fold symmetric product $\text{Sym}^k(\hat{F})$ ($1 \leq k \leq n$), equipped with the product complex structure J , and the holomorphic map $f_{n,k} : \text{Sym}^k(\hat{F}) \rightarrow \mathbb{C}$ defined by $f_{n,k}(\{z_1, \dots, z_k\}) = \pi(z_1) + \dots + \pi(z_k)$.

Proposition 12. $f_{n,k} : \text{Sym}^k(\hat{F}) \rightarrow \mathbb{C}$ is a holomorphic map with isolated non-degenerate critical points (i.e., a Lefschetz fibration); its $\binom{n}{k}$ critical points are the tuples consisting of k distinct points in $\{q_1, \dots, q_n\}$.

Proof. Given $\underline{z} \in \text{Sym}^k(\hat{F})$, denote by z_1, \dots, z_r the distinct elements in the k -tuple \underline{z} , and by k_1, \dots, k_r the multiplicities with which they appear. The tangent space $T_{\underline{z}}\text{Sym}^k(\hat{F})$ decomposes into the direct sum of the $T_{\{z_i, \dots, z_i\}}\text{Sym}^{k_i}(\hat{F})$, and $df_{n,k}(\underline{z})$ splits into the direct sum of the differentials $df_{n,k_i}(\{z_i, \dots, z_i\})$. Thus \underline{z} is a critical point of $f_{n,k}$ if and only if $\{z_i, \dots, z_i\}$ is a critical point of f_{n,k_i} for each $i \in \{1, \dots, r\}$.

By considering the restriction of f_{n,k_i} to the diagonal stratum, we see that $\{z_i, \dots, z_i\}$ cannot be a critical point of f_{n,k_i} unless z_i is a critical point of π . Assume now that z_i is a critical point of π , and pick a local complex coordinate w on \hat{F} near z_i , in which $\pi(w) = w^2 + \text{constant}$. Then a neighborhood of $\{z_i, \dots, z_i\}$

in $\text{Sym}^{k_i}(\hat{F})$ identifies with a neighborhood of the origin in $\text{Sym}^{k_i}(\mathbb{C})$, with coordinates given by the elementary symmetric functions $\sigma_1, \dots, \sigma_{k_i}$. The local model for f_{n,k_i} is then

$$f_{n,k_i}(\{w_1, \dots, w_{k_i}\}) = w_1^2 + \dots + w_{k_i}^2 + \text{constant} = \sigma_1^2 - 2\sigma_2 + \text{constant}.$$

Thus, for $k_i \geq 2$ the point $\{z_i, \dots, z_i\}$ is never a critical point of f_{n,k_i} . We conclude that the only critical points of $f_{n,k}$ are tuples of distinct critical points of π ; moreover these critical points are clearly non-degenerate. \square

For $s \in \mathcal{S}_k^n$, we denote by $Q_s = \{q_i, i \in s\}$ the corresponding critical point of $f_{n,k}$, and by $P_s = \sum_{i \in s} p_i$ the associated critical value.

As in §2, equip $\text{Sym}^k(\hat{F})$ with a Kähler form ω which is of product type away from the diagonal strata, and the associated Kähler metric. This allows us to associate to each critical point Q_s a properly embedded Lagrangian disc \hat{D}_s in $\text{Sym}^k(\hat{F})$ (called *Lefschetz thimble*), namely the set of those points in $f_{n,k}^{-1}(\mathbb{R}_{\geq 0} + P_s)$ for which the gradient flow of $\text{Re } f_{n,k}$ converges to the critical point Q_s . A straightforward calculation shows that $\hat{D}_s = \prod_{i \in s} \hat{\alpha}_i$.

More generally, one can associate a Lefschetz thimble to any properly embedded arc γ connecting P_s to infinity: namely, the set of points in $f_{n,k}^{-1}(\gamma)$ for which symplectic parallel transport converges to the critical point Q_s . We will only consider the case where γ is a straight half-line. Given $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, the thimble associated to the half-line $e^{i\theta}\mathbb{R}_{\geq 0} + P_s$ is again a product $\hat{D}_s(\theta) = \prod_{i \in s} \hat{\alpha}_i(\theta)$, where $\hat{\alpha}_i(\theta)$ is the union of the two lifts of the half-line $e^{i\theta}\mathbb{R}_{\geq 0} + p_j$ through q_j .

3.3.2. A special case of Theorem 1. In the same setting as above, consider the Riemann surface with boundary $F = \pi^{-1}(D^2)$, i.e. the preimage of the unit disc, and let $Z = \pi^{-1}(-1) \subset \partial F$. Let $\alpha_i = \hat{\alpha}_i \cap F$, and $D_s = \hat{D}_s \cap \text{Sym}^k(F) = \prod_{i \in s} \alpha_i$. Then we can reinterpret the partially wrapped Fukaya category $\mathcal{F}(\text{Sym}^k(F), Z)$ and the algebra $\mathcal{A}(\mathbb{F}, k)$ associated to the arcs $\alpha_1, \dots, \alpha_n$ in different terms.

Seidel associates to the Lefschetz fibration $f_{n,k}$ a Fukaya category $\mathcal{F}(f_{n,k})$, whose objects are compact Lagrangian submanifolds of $\text{Sym}^k(\hat{F})$ on one hand, and Lefschetz thimbles associated to admissible arcs connecting a critical value of $f_{n,k}$ to infinity on the other hand [15]. Here we say that an arc is admissible with slope $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ if outside of a compact set it is a half-line of slope θ . (Seidel considers the case of an exact symplectic form, and defines things somewhat differently; however our setting does not pose any significant additional difficulties).

Morphisms between thimbles in $\mathcal{F}(f_{n,k})$ (and compositions thereof) are defined by means of suitable perturbations. Namely, given two admissible arcs γ_0, γ_1 and the corresponding thimbles $D_0, D_1 \subset \text{Sym}^k(\hat{F})$, one sets $\text{hom}_{\mathcal{F}(f_{n,k})}(D_0, D_1) = CF(\tilde{D}_0, \tilde{D}_1)$, where \tilde{D}_0, \tilde{D}_1 are thimbles obtained by suitably perturbing (γ_0, γ_1) to a positive pair $(\tilde{\gamma}_0, \tilde{\gamma}_1)$, i.e. one for which the slopes satisfy $\theta'_0 > \theta'_1$.

Restricting ourselves to the special case of straight half-lines, and observing that for sufficiently small $\theta_0 > \dots > \theta_\ell$ the collections of arcs $\alpha_i(\theta_j) = \hat{\alpha}_i(\theta_j) \cap F$ form a positive sequence in the sense of §3.1, it is not hard to see that we have an

isomorphism of A_∞ -algebras

$$\bigoplus_{s,t \in \mathcal{S}_k^n} \text{hom}_{\mathcal{F}(\text{Sym}^k(F), Z)}(D_s, D_t) \simeq \bigoplus_{s,t \in \mathcal{S}_k^n} \text{hom}_{\mathcal{F}(f_{n,k})}(\hat{D}_s, \hat{D}_t).$$

A key result due to Seidel is the following:

Theorem 13 (Seidel [15], Theorem 18.24). *The A_∞ -category $\mathcal{F}(f_{n,k})$ is generated by the exceptional collection of thimbles \hat{D}_s , $s \in \mathcal{S}_k^n$.*

In other terms, every object of $\mathcal{F}(f_{n,k})$ is quasi-isomorphic to a twisted complex built out of the objects \hat{D}_s , $s \in \mathcal{S}_k^n$.

This implies Theorem 1 in the special case where F is a simple branched cover of the disc with n critical points, Z is the preimage of -1 , and the arcs $\alpha_1, \dots, \alpha_n$ are lifts of half-lines connecting the critical values to the boundary of the disc along the real positive direction. (More precisely, in view of the relation between $\mathcal{F}(f_{n,k})$ and $\mathcal{F}(\text{Sym}^k(F), Z)$, Seidel's result directly implies that the compact objects of $\mathcal{F}(\text{Sym}^k(F), Z)$ are generated by the D_s . On the other hand, arbitrary products of properly embedded arcs cannot be viewed as objects of $\mathcal{F}(f_{n,k})$, but by performing sequences of arc slides we can express them explicitly as iterated mapping cones involving the generators D_s , see below.)

3.3.3. Acceleration functors. Consider a fixed surface F , and two subsets $Z \subseteq Z' \subset \partial F$. Then there exists a natural A_∞ -functor from $\mathcal{F}(\text{Sym}^k(F), Z')$ to $\mathcal{F}(\text{Sym}^k(F), Z)$, called “acceleration functor”. This functor is identity on objects, and in the present case it is simply given by an inclusion of morphism spaces. In general, it is given by the Floer-theoretic continuation maps that arise when comparing the Hamiltonian perturbations used to define morphisms and compositions in $\mathcal{F}(\text{Sym}^k(F), Z')$ and $\mathcal{F}(\text{Sym}^k(F), Z)$.

Consider two products $\Delta = \delta_1 \times \dots \times \delta_k$ and $L = \lambda_1 \times \dots \times \lambda_k$ of disjoint properly embedded arcs in F with boundary in $\partial F \setminus Z'$. Perturbing the arcs $\delta_1, \dots, \delta_k$ and $\lambda_1, \dots, \lambda_k$ near ∂F if needed (without crossing Z'), we can assume that the pair (Δ, L) is positive with respect to Z' . On the other hand, achieving positivity with respect to the smaller subset Z may require a further perturbation of the arcs δ_i (resp. λ_i) in the positive (resp. negative) direction along ∂F , to obtain product Lagrangians $\tilde{\Delta} = \tilde{\delta}_1 \times \dots \times \tilde{\delta}_k$ and $\tilde{L} = \tilde{\lambda}_1 \times \dots \times \tilde{\lambda}_k$. This perturbation can be performed in such a way as to only *create* new intersection points. The local picture is as shown on Figure 3. The key observation is that none of the intersection points created in the isotopy can be the outgoing end of a holomorphic strip in $\text{Sym}^k(F)$ with boundary on $\tilde{\Delta} \cup \tilde{L}$ and whose incoming end is a previously existing intersection point (i.e., one that arises by deforming a point of $\Delta \cap L$). Indeed, considering Figure 3 right, locally the projection of this holomorphic strip to F would cover one of the two regions labelled I and II; but then by the maximum principle it would need to hit ∂F , which is not allowed. This implies that $CF(\Delta, L)$ is naturally a subcomplex of $CF(\tilde{\Delta}, \tilde{L})$. The same argument also holds for products and higher compositions, ensuring that the acceleration functor is well-defined.

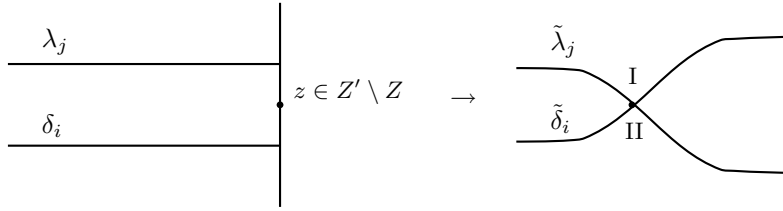


Figure 3. Perturbation and the acceleration functor

In particular, given a collection $\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$ of disjoint properly embedded arcs in F , and setting $\mathbb{F} = (F, Z, \underline{\alpha})$ and $\mathbb{F}' = (F, Z', \underline{\alpha}')$, we obtain that $\mathcal{A}(\mathbb{F}', k)$ is naturally an A_∞ -subalgebra of $\mathcal{A}(\mathbb{F}, k)$ for all k .

Finally, one easily checks that the acceleration functor is unital (at least on cohomology), and surjective on (isomorphism classes of) objects. Hence, if the $\binom{n}{k}$ objects $D_s = \prod_{i \in s} \alpha_i$ ($s \in \mathcal{S}_k^n$) generate $\mathcal{F}(\text{Sym}^k(F), Z')$, then they also generate $\mathcal{F}(\text{Sym}^k(F), Z)$. (Indeed, the assumption means that any object L of $\mathcal{F}(\text{Sym}^k(F), Z')$ is quasi-isomorphic to a twisted complex built out of the D_s ; since A_∞ -functors are exact, this implies that L is also quasi-isomorphic to the corresponding twisted complex in $\mathcal{F}(\text{Sym}^k(F), Z)$).

3.3.4. Eliminating generators by arc slides. We now consider a general decorated surface $\mathbb{F} = (F, Z, \underline{\alpha})$. The arcs $\alpha_1, \dots, \alpha_n$ on F might not be a full set of Lefschetz thimbles for any simple branched covering map, but they are always a subset of the thimbles of a more complicated covering (with m critical points, $m \geq n$). Namely, after a suitable deformation (which does not affect the symplectic topology of the completed symmetric product $\text{Sym}^k(\hat{F})$), we can always assume that F projects to the disc by a simple branched covering map π with critical values p_1, \dots, p_m , in such a way that the arcs $\alpha_1, \dots, \alpha_n$ are lifts of n of the half-lines $\mathbb{R}_{\geq 0} + p_j$, while each point of Z projects to -1 . Hence, taking the remaining critical values of π and elements of $\pi^{-1}(-1)$ into account, there exists a subset $Z' \supseteq Z$ of ∂F , and a collection $\underline{\alpha}'$ of $m \geq n$ disjoint properly embedded arcs (including the α_i), such that $\mathbb{F}' = (F, Z', \underline{\alpha}')$ is as in §3.3.2. Then, as seen above, the partially wrapped Fukaya category $\mathcal{F}(\text{Sym}^k(F), Z')$ is generated by the $\binom{m}{k}$ product objects $D'_s = \prod_{i \in s} \alpha'_i$ ($s \in \mathcal{S}_k^m$).

Moreover, by considering the acceleration functor as in §3.3.3, we conclude that $\mathcal{F}(\text{Sym}^k(F), Z)$ is also generated by the objects D'_s , $s \in \mathcal{S}_k^m$. Thus, Theorem 1 follows if, assuming that each component of $F \setminus (\alpha_1 \cup \dots \cup \alpha_n)$ is a disc containing at most one point of Z , we can show that the $\binom{m}{k} - \binom{n}{k}$ additional objects we have introduced can be expressed in terms of the others. This is done by eliminating the additional arcs α'_i one at a time.

Consider $k+1$ disjoint properly embedded arcs $\lambda_1, \dots, \lambda_k, \lambda'_1$ in F , with boundary in $\partial F \setminus Z$, and such that an end point of λ'_1 lies immediately after an end point of λ_1 along a component of $\partial F \setminus Z$. Let $\tilde{\lambda}'_1$ be the arc obtained by sliding λ_1 along λ'_1 . Finally, denote by $\tilde{\lambda}_1, \dots, \tilde{\lambda}_k$ a collection of arcs obtained by slightly perturb-

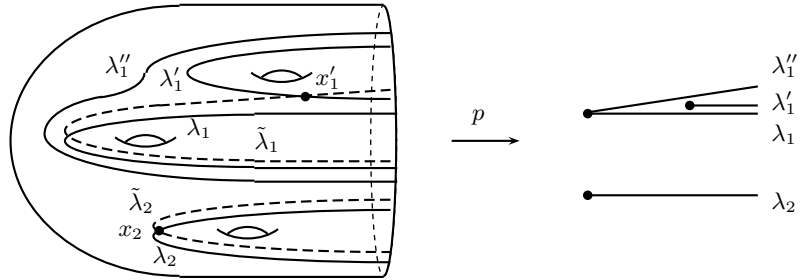


Figure 4. Sliding λ_1 along λ'_1 , and the auxiliary covering p

ing $\lambda_1, \dots, \lambda_k$ in the positive direction, with each $\tilde{\lambda}_i$ intersecting λ_i in a single point $x_i \in U$, and $\tilde{\lambda}_1$ intersecting λ'_1 in a single point x'_1 which lies near the boundary; see Figure 4. Let $L = \lambda_1 \times \dots \times \lambda_k$, $L' = \lambda'_1 \times \lambda_2 \times \dots \times \lambda_k$, and $L'' = \lambda'_1 \times \lambda_2 \times \dots \times \lambda_k$. Then the point $(x'_1, x_2, \dots, x_k) \in (\tilde{\lambda}_1 \times \dots \times \tilde{\lambda}_k) \cap (\lambda'_1 \times \lambda_2 \times \dots \times \lambda_k)$ determines (via the appropriate continuation map between Floer complexes, to account for the need to further perturb L) an element of $\text{hom}(L, L')$, which we call u . The following result is essentially Lemma 5.2 of [2].

Lemma 14 ([2]). *In the A_∞ -category of twisted complexes $\text{Tw}\mathcal{F}(\text{Sym}^k(F), Z)$, L'' is quasi-isomorphic to the mapping cone of u .*

The main idea is to consider an auxiliary simple branched covering $p: \hat{F} \rightarrow \mathbb{C}$ for which the arcs $\lambda_1, \lambda'_1, \dots, \lambda_k$ are Lefschetz thimbles (i.e., lifts of half-lines), with the critical value for λ'_1 lying immediately next to that for λ_1 and so that the monodromies at the corresponding critical values are transpositions with one common index (see Figure 4 right). The objects L, L', L'' can be viewed as Lefschetz thimbles for the Lefschetz fibration induced by p on the symmetric product; in the corresponding Fukaya category, the statement that $L'' \simeq \text{Cone}(u)$ follows from a general result of Seidel [15, Proposition 18.23]. The lemma then follows from exactness of the relevant acceleration functor. See §5 of [2] for details.

The other useful fact is that sliding one factor of L over another factor of L only affects L by a Hamiltonian isotopy. For instance, in the above situation, $\lambda_1 \times \lambda'_1 \times \lambda_3 \times \dots \times \lambda_k$ and $\lambda'_1 \times \lambda_1 \times \lambda_3 \times \dots \times \lambda_k$ are Hamiltonian isotopic. (This is an easy consequence of the main result in [11]).

Returning to the collection of arcs α' on the surface F , assume that α'_m can be erased without losing the property that every component of the complement is a disc carrying at most one point of Z . Then one of the connected components of $F \setminus (\alpha'_1 \cup \dots \cup \alpha'_m)$ is a disc Δ which contains no point of Z , and whose boundary consists of portions of ∂F and the arcs $\alpha'_m, \alpha'_{i_1}, \dots, \alpha'_{i_r}$ (with i_1, \dots, i_r distinct from m , but not necessarily pairwise distinct) in that order. Then the arc obtained by sliding α'_{i_1} successively over $\alpha'_{i_2}, \dots, \alpha'_{i_r}$ is isotopic to α'_m . Hence, by Lemma 14, for $m \in s$ the object D'_s can be expressed as a twisted complex built from the objects D'_{s_j} , where $s_j = (s \cup \{i_j\}) \setminus \{m\}$, for $j \in \{1, \dots, r\}$ such that $i_j \notin s$.

4. Yoneda embedding and invariants of bordered 3-manifolds

Let $\mathbb{F} = (F, Z, \underline{\alpha})$ be a decorated surface, and assume that every component of $F \setminus (\bigcup \alpha_i)$ is a disc carrying at most one point of Z . By Theorem 1 the partially wrapped Fukaya category $\mathcal{F}(\mathrm{Sym}^k(F), Z)$ is generated by the product objects D_s , $s \in \mathcal{S}_k^n$. In fact Theorem 1 continues to hold if we consider the extended category $\mathcal{F}^\#(\mathrm{Sym}^k(F), Z)$ instead of $\mathcal{F}(\mathrm{Sym}^k(F), Z)$; see Proposition 6.3 of [2]. (The key point is that the only generalized Lagrangians we consider are compactly supported in the interior of $\mathrm{Sym}^k(F)$, and Seidel's argument for generation of compact objects by Lefschetz thimbles still applies to them.)

To each object \mathbb{T} of $\mathcal{F}^\#(\mathrm{Sym}^k(F), Z)$ we can associate a right A_∞ -module over the algebra $\mathcal{A} = \mathcal{A}(\mathbb{F}, k)$,

$$\mathcal{Y}(\mathbb{T}) = \mathcal{Y}^r(\mathbb{T}) = \bigoplus_{s \in \mathcal{S}_k^n} \mathrm{hom}(\mathbb{T}, D_s) \in \mathrm{mod}\text{-}\mathcal{A},$$

where the module maps $m_\ell : \mathcal{Y}(\mathbb{T}) \otimes \mathcal{A}^{\otimes(\ell-1)} \rightarrow \mathcal{Y}(\mathbb{T})$ are defined by products and higher compositions in $\mathcal{F}^\#(\mathrm{Sym}^k(F), Z)$. Moreover, given two objects $\mathbb{T}_0, \mathbb{T}_1$, compositions in the partially wrapped Fukaya category yield a natural map from $\mathrm{hom}(\mathbb{T}_0, \mathbb{T}_1)$ to $\mathrm{hom}_{\mathrm{mod}\text{-}\mathcal{A}}(\mathcal{Y}(\mathbb{T}_1), \mathcal{Y}(\mathbb{T}_0))$, as well as higher order maps. Thus, we obtain a contravariant A_∞ -functor $\mathcal{Y} : \mathcal{F}^\#(\mathrm{Sym}^k(F), Z) \rightarrow \mathrm{mod}\text{-}\mathcal{A}(\mathbb{F}, k)$: the *right Yoneda embedding*.

Proposition 15. *Under the assumptions of Theorem 1, \mathcal{Y} is a cohomologically full and faithful (contravariant) embedding.*

Indeed, the general Yoneda embedding into $\mathrm{mod}\text{-}\mathcal{F}^\#(\mathrm{Sym}^k(F), Z)$ is cohomologically full and faithful (see e.g. [15, Corollary 2.13]), while Theorem 1 (or rather its analogue for the extended Fukaya category) implies that the natural functor from $\mathrm{mod}\text{-}\mathcal{F}^\#(\mathrm{Sym}^k(F), Z)$ to $\mathrm{mod}\text{-}\mathcal{A}(\mathbb{F}, k)$ given by restricting an arbitrary A_∞ -module to the subset of objects $\{D_s, s \in \mathcal{S}_k^n\}$ is an equivalence.

We can similarly consider the *left Yoneda embedding* to left A_∞ -modules over $\mathcal{A}(\mathbb{F}, k)$, namely the (covariant) A_∞ -functor $\mathcal{Y}^\ell : \mathcal{F}^\#(\mathrm{Sym}^k(F), Z) \rightarrow \mathcal{A}(\mathbb{F}, k)\text{-mod}$ which sends the object \mathbb{T} to $\mathcal{Y}^\ell(\mathbb{T}) = \bigoplus_{s \in \mathcal{S}_k^n} \mathrm{hom}(D_s, \mathbb{T})$.

Lemma 16. *Denote by $-\mathbb{F} = (-F, Z, \underline{\alpha})$ the decorated surface obtained by orientation reversal. Then $\mathcal{A}(-\mathbb{F}, k)$ is isomorphic to the opposite A_∞ -algebra $\mathcal{A}(\mathbb{F}, k)^{op}$.*

Proof. Given $s_0, \dots, s_\ell \in \mathcal{S}_k^n$, and any positive perturbation $(\tilde{D}_{s_0}, \dots, \tilde{D}_{s_\ell})$ of the sequence $(D_{s_0}, \dots, D_{s_\ell})$ in $\mathrm{Sym}^k(F)$ relatively to Z , the reversed sequence $(\tilde{D}_{s_\ell}, \dots, \tilde{D}_{s_0})$ is a positive perturbation of $(D_{s_\ell}, \dots, D_{s_0})$ in $\mathrm{Sym}^k(-F)$. Thus, the holomorphic discs in $\mathrm{Sym}^k(-F)$ which contribute to the product operation $m_\ell : \mathrm{hom}(D_{s_\ell}, D_{s_{\ell-1}}) \otimes \dots \otimes \mathrm{hom}(D_{s_1}, D_{s_0}) \rightarrow \mathrm{hom}(D_{s_\ell}, D_{s_0})$ in $\mathcal{A}(-\mathbb{F}, k)$ are exactly the complex conjugates of the holomorphic discs in $\mathrm{Sym}^k(F)$ which contribute to $m_\ell : \mathrm{hom}(D_{s_0}, D_{s_1}) \otimes \dots \otimes \mathrm{hom}(D_{s_{\ell-1}}, D_{s_\ell}) \rightarrow \mathrm{hom}(D_{s_0}, D_{s_\ell})$ in $\mathcal{A}(\mathbb{F}, k)$. \square

Hence, left A_∞ -modules over $\mathcal{A} = \mathcal{A}(\mathbb{F}, k)$ can be interchangeably viewed as right A_∞ -modules over $\mathcal{A}^{op} = \mathcal{A}(-\mathbb{F}, k)$; more specifically, given a generalized Lagrangian \mathbb{T} in $\text{Sym}^k(F)$ and its conjugate $-\mathbb{T}$ in $\text{Sym}^k(-F)$, the left Yoneda module $\mathcal{Y}^\ell(\mathbb{T}) \in \mathcal{A}\text{-mod}$ is the same as the right Yoneda module $\mathcal{Y}^r(-\mathbb{T}) \in \text{mod-}\mathcal{A}^{op}$.

Moreover, the left and right Yoneda embeddings are dual to each other:

Lemma 17. *For any object \mathbb{T} , the modules $\mathcal{Y}^\ell(\mathbb{T}) \in \mathcal{A}\text{-mod}$ and $\mathcal{Y}^r(\mathbb{T}) \in \text{mod-}\mathcal{A}$ satisfy $\mathcal{Y}^r(\mathbb{T}) \simeq \text{hom}_{\mathcal{A}\text{-mod}}(\mathcal{Y}^\ell(\mathbb{T}), \mathcal{A})$ and $\mathcal{Y}^\ell(\mathbb{T}) \simeq \text{hom}_{\text{mod-}\mathcal{A}}(\mathcal{Y}^r(\mathbb{T}), \mathcal{A})$ (where \mathcal{A} is viewed as an A_∞ -bimodule over itself).*

Proof. By definition, $\mathcal{Y}^r(\mathbb{T}) = \text{hom}(\mathbb{T}, \bigoplus_s D_s)$ (working in an additive enlargement of $\mathcal{F}^\#(\text{Sym}^k(F), Z)$), with the right A_∞ -module structure coming from right composition (and higher products) with endomorphisms of $\bigoplus_s D_s$. However, the left Yoneda embedding functor is full and faithful, and maps \mathbb{T} to $\mathcal{Y}^\ell(\mathbb{T})$ and $\bigoplus_s D_s$ to \mathcal{A} . Hence, as chain complexes $\text{hom}(\mathbb{T}, \bigoplus_s D_s) \simeq \text{hom}_{\mathcal{A}\text{-mod}}(\mathcal{Y}^\ell(\mathbb{T}), \mathcal{A})$. Moreover this quasi-isomorphism is compatible with the right module structures (by functoriality of the left Yoneda embedding). The other statement is proved similarly, by applying the right Yoneda functor (contravariant, full and faithful) to prove that $\mathcal{Y}^\ell(\mathbb{T}) = \text{hom}(\bigoplus_s D_s, \mathbb{T}) \simeq \text{hom}_{\text{mod-}\mathcal{A}}(\mathcal{Y}^r(\mathbb{T}), \mathcal{A})$. \square

All the ingredients are now in place for the proof of Theorem 4 (and other similar pairing results). Consider as in the introduction a closed 3-manifold Y obtained by gluing two 3-manifolds Y_1, Y_2 with $\partial Y_1 = -\partial Y_2 = F \cup D^2$ along their common boundary, and equip the surface F with boundary marked points Z and a collection $\underline{\alpha}$ of disjoint properly embedded arcs such that the decorated surface $\mathbb{F} = (F, Z, \underline{\alpha})$ satisfies the assumption of Theorem 1.

Remark. The natural choice in view of Lipshitz-Ozsváth-Thurston's work on bordered Heegaard-Floer homology [6] is to equip F with a single marked point and a collection of $2g$ arcs that decompose it into a single disc, e.g. as in Figure 1. However, one could also equip F with *two* boundary marked points and $2g + 1$ arcs, by viewing F as a double cover of the unit disc with $2g + 1$ branch points and proceeding as in §3.3.2; see Figure 5. While this yields a larger generating set, with $\binom{2g+1}{k}$ objects instead of $\binom{2g}{k}$, the resulting algebra remains combinatorial in nature (by Proposition 11) and it is more familiar from the perspective of symplectic geometry, since we are now dealing with the Fukaya category of a Lefschetz

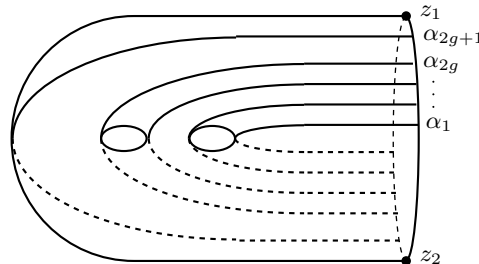


Figure 5. Decorating F with two marked points and $2g + 1$ arcs

fibration on the symmetric product. Among other nice features, the generators are exceptional objects, and the algebra is directed.

Let us now return to our main argument. As explained in §2.3, the work of Lekili and Perutz [4] associates to the 3-manifolds Y_1 and $-Y_2$ (viewed as sutured cobordisms from D^2 to F) two generalized Lagrangian submanifolds \mathbb{T}_{Y_1} and \mathbb{T}_{-Y_2} of $\text{Sym}^g(F)$, with the property that $\widehat{CF}(Y)$ is quasi-isomorphic to $\text{hom}_{\mathcal{F}^\#(\text{Sym}^g(F))}(\mathbb{T}_{Y_1}, \mathbb{T}_{-Y_2})$. However, by Proposition 15 we have

$$\text{hom}_{\mathcal{F}^\#(\text{Sym}^g(F))}(\mathbb{T}_{Y_1}, \mathbb{T}_{-Y_2}) \simeq \text{hom}_{\text{mod-}\mathcal{A}(\mathbb{F}, g)}(\mathcal{Y}(\mathbb{T}_{-Y_2}), \mathcal{Y}(\mathbb{T}_{Y_1}))$$

where $\mathcal{Y} = \mathcal{Y}^r$ denotes the right Yoneda embedding functor. Moreover, using Lemma 17, and setting $\mathcal{A} = \mathcal{A}(\mathbb{F}, g)$, we have:

$$\begin{aligned} \mathcal{Y}^r(\mathbb{T}_{Y_1}) \otimes_{\mathcal{A}} \mathcal{Y}^\ell(\mathbb{T}_{-Y_2}) &\simeq \mathcal{Y}^r(\mathbb{T}_{Y_1}) \otimes_{\mathcal{A}} \text{hom}_{\text{mod-}\mathcal{A}}(\mathcal{Y}^r(\mathbb{T}_{-Y_2}), \mathcal{A}) \\ &\simeq \text{hom}_{\text{mod-}\mathcal{A}}(\mathcal{Y}^r(\mathbb{T}_{-Y_2}), \mathcal{Y}^r(\mathbb{T}_{Y_1}) \otimes_{\mathcal{A}} \mathcal{A}) \\ &\simeq \text{hom}_{\text{mod-}\mathcal{A}}(\mathcal{Y}^r(\mathbb{T}_{-Y_2}), \mathcal{Y}^r(\mathbb{T}_{Y_1})). \end{aligned}$$

Finally, by the discussion after Lemma 16, we can identify the left $\mathcal{A}(\mathbb{F}, g)$ -module $\mathcal{Y}^\ell(\mathbb{T}_{-Y_2})$ with the right module $\mathcal{Y}^r(\mathbb{T}_{Y_2}) \in \text{mod-}\mathcal{A}(-\mathbb{F}, g)$. This completes the proof of Theorem 4.

Turning to the case of more general cobordisms, recall that the construction of Lekili and Perutz associates to a sutured manifold Y with $\partial Y = (-F_-) \cup F_+$ a generalized Lagrangian correspondence \mathbb{T}_Y from $\text{Sym}^{k_-}(F_-)$ to $\text{Sym}^{k_+}(F_+)$ (where $k_+ - k_- = g(F_+) - g(F_-)$), i.e. an object of $\mathcal{F}^\#(\text{Sym}^{k_-}(-F_-) \times \text{Sym}^{k_+}(F_+))$.

Equip the surfaces F_- and F_+ with sets of boundary marked points Z_\pm and two collections $\underline{\alpha}_\pm$ of properly embedded arcs such that the decorated surfaces $\mathbb{F}_\pm = (F_\pm, Z_\pm, \underline{\alpha}_\pm)$ satisfy the assumption of Theorem 1. Considering products of k_\pm of the arcs in $\underline{\alpha}_\pm$, we have two collections of product Lagrangian submanifolds $D_{\pm, s}$ ($s \in \mathcal{S}_\pm$) in $\text{Sym}^{k_\pm}(F_\pm)$. By a straightforward generalization of Theorem 1, the partially wrapped Fukaya category $\mathcal{F}^\#(\text{Sym}^{k_-}(-F_-) \times \text{Sym}^{k_+}(F_+), Z_- \sqcup Z_+)$ is generated by the product objects $(-D_{-, s}) \times D_{+, t}$ for $(s, t) \in \mathcal{S}_- \times \mathcal{S}_+$. Indeed, $\text{Sym}^{k_-}(-F_-) \times \text{Sym}^{k_+}(F_+)$ is a connected component of $\text{Sym}^{k_- + k_+}((-F_-) \sqcup F_+)$, and the proof of Theorem 1 applies without modification to the disconnected decorated surface $(-\mathbb{F}_-) \sqcup \mathbb{F}_+ = ((-F_-) \sqcup F_+, Z_- \sqcup Z_+, \underline{\alpha}_- \sqcup \underline{\alpha}_+)$. Hence, as before, the Yoneda construction

$$\mathcal{Y}(\mathbb{T}_Y) = \bigoplus_{(s, t) \in \mathcal{S}_- \times \mathcal{S}_+} \text{hom}(\mathbb{T}_Y, (-D_{-, s}) \times D_{+, t})$$

defines a cohomologically full and faithful embedding into the category of right A_∞ -bimodules over $\mathcal{A}(-\mathbb{F}_-, k_-)$ and $\mathcal{A}(\mathbb{F}_+, k_+)$, or equivalently, the category of A_∞ -bimodules $\mathcal{A}(\mathbb{F}_-, k_-)\text{-mod-}\mathcal{A}(\mathbb{F}_+, k_+)$. This property is the key ingredient that makes it possible to relate compositions of generalized Lagrangian correspondences (i.e., gluing of sutured cobordisms) to algebraic operations on A_∞ -bimodules, as in Conjecture 3 for instance.

5. Relation to bordered Heegaard-Floer homology

Consider a sutured 3-manifold Y , with $\partial Y = (-F_-) \cup (\Gamma \times [0, 1]) \cup F_+$, and pick decorations $\mathbb{F}_\pm = (F_\pm, Z_\pm, \underline{\alpha}_\pm)$ of F_\pm . Assume for simplicity that $Z_+ = Z_-$. Denote by g_\pm the genus of F_\pm , and by n_\pm the number of arcs in $\underline{\alpha}_\pm$. Choose a Morse function $f : Y \rightarrow [0, 1]$ with index 1 and 2 critical points only, such that $f^{-1}(1) = F_-$ and $f^{-1}(0) = F_+$. Assume that all the index 1 critical points lie in $f^{-1}((0, \frac{1}{2}))$ and all the index 2 critical points lie in $f^{-1}((\frac{1}{2}, 1))$. Also pick a gradient-like vector field for f , tangent to the boundary along $\Gamma \times [0, 1]$, and equip the level sets of f with complex structures such that the gradient flow induces biholomorphisms away from the critical locus. The above data determines a *bordered Heegaard diagram* on the surface $\Sigma = f^{-1}(\frac{1}{2})$ of genus $\bar{g} = g(\Sigma)$, consisting of:

- $\bar{g} - g_+$ simple closed curves $\alpha_1^c, \dots, \alpha_{\bar{g}-g_+}^c$, where α_i^c is the set of points of Σ from which the downwards gradient flow converges to the i -th index 1 critical point;
- n_+ properly embedded arcs $\alpha_1^a, \dots, \alpha_{n_+}^a$, where α_i^a is the set of points of Σ from which the downwards gradient flow ends at a point of $\alpha_{+,i} \subset F_+$;
- $\bar{g} - g_-$ simple closed curves $\beta_1^c, \dots, \beta_{\bar{g}-g_-}^c$, where β_i^c is the set of points of Σ from which the upwards gradient flow converges to the i -th index 2 critical point;
- n_- properly embedded arcs $\beta_1^a, \dots, \beta_{n_-}^a$, where β_i^a is the set of points of Σ from which the upwards gradient flow ends at a point of $\alpha_{-,i} \subset F_-$;
- a finite set Z of boundary marked points (which match with Z_\pm under the gradient flow).

Given integers \bar{k}, k_+, k_- satisfying $\bar{k} - \bar{g} = k_+ - g_+ = k_- - g_-$, we can view the generalized Lagrangian correspondence \mathbb{T}_Y associated to Y as the composition of the correspondence $T_\beta \subset \text{Sym}^{k_-}(-F_-) \times \text{Sym}^{\bar{k}}(\Sigma)$ determined by $f^{-1}([\frac{1}{2}, 1])$ and the correspondence $T_\alpha \subset \text{Sym}^{\bar{k}}(-\Sigma) \times \text{Sym}^{k_+}(F_+)$ determined by $f^{-1}([0, \frac{1}{2}])$.

The A_∞ -bimodule $\mathcal{Y}(\mathbb{T}_Y) \in \mathcal{A}(\mathbb{F}_-, k_-)\text{-mod-}\mathcal{A}(\mathbb{F}_+, k_+)$ associated to Y can then be understood entirely in terms of the symmetric product $\text{Sym}^{\bar{k}}(\Sigma)$. Namely, denote by $\tilde{\mathcal{F}}^\# = \tilde{\mathcal{F}}^\#(\text{Sym}^{\bar{k}}(\Sigma), Z)$ a partially wrapped Fukaya category defined similarly to the construction in §3, except we also allow objects which are products of mutually disjoint simple closed curves and properly embedded arcs in Σ .

The Lagrangian correspondences $-T_\alpha$ and T_β induce A_∞ -functors Φ_α and Φ_β from $\mathcal{F}^\#(\text{Sym}^{k_\pm}(F_\pm), Z_\pm)$ to $\tilde{\mathcal{F}}^\#$. Considering the product Lagrangians $D_{-,s}$ for $s \in \mathcal{S}_- = \mathcal{S}_{k_-}^{n_-}$, the description of the geometry of the correspondence T_β away from the diagonal [10] (or the result of [4]) implies that $\Phi_\beta(D_{-,s})$, i.e., the composition of $D_{-,s}$ with the correspondence T_β , is Hamiltonian isotopic to

$$\Delta_{\beta,s} = \prod_{i \in s} \beta_i^a \times \prod_{j=1}^{\bar{g}-g_-} \beta_j^c \subset \text{Sym}^{\bar{k}}(\Sigma).$$

Similarly, for $t \in \mathcal{S}_+ = \mathcal{S}_{k_+}^{n_+}$ the image of $D_{+,t}$ under the correspondence $(-T_\alpha)$ is Hamiltonian isotopic to the product

$$\Delta_{\alpha,t} = \prod_{i \in t} \alpha_i^a \times \prod_{j=1}^{\bar{g}-g_+} \alpha_j^c \subset \text{Sym}^{\bar{k}}(\Sigma).$$

This implies the following result:

Proposition 18. *The A_∞ -bimodule $\mathcal{Y}(\mathbb{T}_Y) \in \mathcal{A}(\mathbb{F}_-, k_-)\text{-mod-}\mathcal{A}(\mathbb{F}_+, k_+)$ is quasi-isomorphic to $\bigoplus_{s,t} \text{hom}_{\mathcal{F}\#}(\Delta_{\beta,s}, \Delta_{\alpha,t})$.*

To clarify this statement, observe that Φ_α induces an A_∞ -homomorphism from $\mathcal{A}(\mathbb{F}_+, k_+) = \bigoplus_{s,t} \text{hom}(D_{+,s}, D_{+,t})$ to $\mathcal{A}_\alpha = \bigoplus_{s,t} \text{hom}_{\mathcal{F}\#}(\Delta_{\alpha,s}, \Delta_{\alpha,t})$. In fact, suitable choices in the construction ensure that $\mathcal{A}_\alpha \simeq \mathcal{A}(\mathbb{F}_+, k_+) \otimes H^*(T^{\bar{g}-g_+}, \mathbb{Z}_2)$ and the map from $\mathcal{A}(\mathbb{F}_+, k_+)$ to \mathcal{A}_α is simply given by $x \mapsto x \otimes 1$. In any case, via Φ_α we can view any right A_∞ -module over \mathcal{A}_α as a right A_∞ -module over $\mathcal{A}(\mathbb{F}_+, k_+)$. Similarly, Φ_β induces an A_∞ -homomorphism from $\mathcal{A}(\mathbb{F}_-, k_-)$ to $\mathcal{A}_\beta = \bigoplus_{s,t} \text{hom}_{\mathcal{F}\#}(\Delta_{\beta,s}, \Delta_{\beta,t})$, through which any left A_∞ -module over \mathcal{A}_β can be viewed as a left A_∞ -module over $\mathcal{A}(\mathbb{F}_-, k_-)$.

With this understood, Proposition 18 essentially follows from the fact that the A_∞ -functors induced by the correspondences T_α and $(-T_\alpha)$ on one hand, and T_β and $(-T_\beta)$ on the other hand, are adjoint to each other; see Proposition 6.6 in [2] for the case of A_∞ -modules.

The case where one of k_\pm vanishes, say $k_- = 0$, is of particular interest; then the β -arcs play no role whatsoever, and we only need to consider the product torus $T_\beta = \beta_1^c \times \cdots \times \beta_{\bar{g}-g_-}^c \subset \text{Sym}^{\bar{k}}(\Sigma)$. This happens for instance when F_- is a disc, i.e. when Y is a 3-manifold with boundary $\partial Y = F_+ \cup D^2$ viewed as a sutured cobordism from D^2 to F_+ . (This corresponds to the situation considered in [6]; in this case we have $\bar{k} = \bar{g}$ and $k_+ = g_+$).

In this situation, the statement of Proposition 18 becomes that the right A_∞ -module $\mathcal{Y}(\mathbb{T}_Y) \in \text{mod-}\mathcal{A}(\mathbb{F}_+, k_+)$ is quasi-isomorphic to $\bigoplus_{t \in \mathcal{S}_+} \text{hom}_{\mathcal{F}\#}(T_\beta, \Delta_{\alpha,t})$. Then we have the following result (Proposition 6.5 of [2]):

Proposition 19. *The right A_∞ -modules over $\mathcal{A}(\mathbb{F}_+, k_+)$ constructed by Yoneda embedding, $\mathcal{Y}(\mathbb{T}_Y) \simeq \bigoplus_{t \in \mathcal{S}_+} \text{hom}_{\mathcal{F}\#}(T_\beta, \Delta_{\alpha,t})$, and by bordered Heegaard-Floer homology, $\widehat{CFA}(Y)$, are quasi-isomorphic.*

The fact that $\bigoplus_t \text{hom}_{\mathcal{F}\#}(T_\beta, \Delta_{\alpha,t})$ and $\widehat{CFA}(Y)$ are quasi-isomorphic (in fact isomorphic) as chain complexes is a straightforward consequence of the definitions. Comparing the module structures requires a comparison of the moduli spaces of holomorphic curves which determine the module maps; this can be done via a neck-stretching argument, see [2, Proposition 6.5].

Remark. Another special case worth mentioning is when $k_+ = k_- = 0$, which requires the sutured manifold Y to be balanced in the sense of [3]. Then we can discard all the arcs from the Heegaard diagram, and $\mathcal{Y}(\mathbb{T}_Y) \simeq \text{hom}_{\mathcal{F}\#}(T_\beta, T_\alpha)$ is simply the chain complex which defines the sutured Floer homology of [3]. In this

sense, bordered Heegaard-Floer homology and our constructions can be viewed as natural generalizations of Juhász’s sutured Floer homology. (An even greater level of generality is considered in [18].)

In light of the relation between $\mathcal{Y}(\mathbb{T}_Y)$ and $\widehat{CFA}(Y)$, it is interesting to compare Theorem 4 with the pairing theorem obtained by Lipshitz, Ozsváth and Thurston for bordered Heegaard-Floer homology [6]. In particular, a side-by-side comparison suggests that the modules $\widehat{CFA}(Y)$ and $\widehat{CFD}(Y)$ might be quasi-isomorphic.

Another surprising aspect, about which we can only offer speculation, is the seemingly different manners in which bimodules arise in the two stories. In our case, bimodules arise from sutured 3-manifolds viewed as cobordisms between decorated surfaces, i.e. from bordered Heegaard diagrams where both α - and β -arcs are simultaneously present; and pairing results arise from “top-to-bottom” stacking of cobordisms. On the other hand, the work of Lipshitz, Ozsváth and Thurston [6, 7] provides a different construction of bimodules associated to cobordisms between decorated surfaces, involving diagrams in which there are no β -arcs; and pairing results arise from “side-by-side” gluing of bordered Heegaard diagrams.

As a possible way to understand “side-by-side” gluing in our framework, observe that given two decorated surfaces $\mathbb{F}_i = (F_i, Z_i, \underline{\alpha}_i)$ for $i = 1, 2$, and given two points $z_1 \in Z_1$ and $z_2 \in Z_2$, we can form the *boundary connected sum* $F = F_1 \cup_{\partial} F_2$ of F_1 and F_2 by attaching a 1-handle (i.e., a band) to small intervals of ∂F_1 and ∂F_2 containing z_1 and z_2 respectively. The surface F can be equipped with the set of marked points $Z = (Z_1 \setminus \{z_1\}) \cup (Z_2 \setminus \{z_2\}) \cup \{z_-, z_+\}$, where z_- and z_+ lie on either side of the connecting handle, and the collection of properly embedded arcs $\underline{\alpha} = \underline{\alpha}_1 \cup \underline{\alpha}_2$. Assume moreover that \mathbb{F}_1 and \mathbb{F}_2 satisfy the conditions of Proposition 11, so that the associated algebras are honest differential algebras. Denoting by \mathbb{F} the decorated surface $(F, Z, \underline{\alpha})$, it is then easy to check that $\mathcal{A}(\mathbb{F}, k) \simeq \bigoplus_{k_1+k_2=k} \mathcal{A}(\mathbb{F}_1, k_1) \otimes \mathcal{A}(\mathbb{F}_2, k_2)$.

Now, given two 3-manifolds Y_1, Y_2 with boundary $\partial Y_i \simeq F_i \cup D^2$, we can form their boundary connected sum $Y = Y_1 \cup_{\partial} Y_2$ by attaching a 1-handle at the points z_1, z_2 ; then $\partial Y = F \cup D^2$, and the bordered Heegaard diagram representing Y is simply the boundary connected sum of the bordered Heegaard diagrams representing Y_1 and Y_2 . Accordingly, the right A_{∞} -module associated to Y is the tensor product (over the ground field \mathbb{Z}_2 !) of the right A_{∞} -modules associated to Y_1 and Y_2 . In the case where $\mathbb{F}_1 \simeq -\mathbb{F}_2$, we can glue a standard handlebody to Y in order to obtain a closed 3-manifold \bar{Y} , namely the result of gluing Y_1 and Y_2 along their entire boundaries rather than just at small discs near the points z_1, z_2 . However, because the decorated surface \mathbb{F} never satisfies the assumption of Theorem 1 (the two new marked points z_{\pm} lie in the same component), the Yoneda functor to A_{∞} -modules over $\mathcal{A}(\mathbb{F}, g)$ is not guaranteed to be full and faithful, so our gluing result does not apply.

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