HOMOLOGICAL MIRROR SYMMETRY FOR PUNCTURED SPHERES

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ABSTRACT. We prove that the wrapped Fukaya category of a punctured sphere (S^2 with an arbitrary number of points removed) is equivalent to the triangulated category of singularities of a mirror Landau-Ginzburg model, proving one side of the homological mirror symmetry conjecture in this case. By investigating fractional gradings on these categories, we conclude that cyclic covers on the symplectic side are mirror to orbifold quotients of the Landau-Ginzburg model.

1. INTRODUCTION

1.1. **Background.** In its original formulation, Kontsevich's celebrated homological mirror symmetry conjecture [25] concerns mirror pairs of Calabi-Yau varieties, for which it predicts an equivalence between the derived category of coherent sheaves of one variety and the derived Fukaya category of the other. This conjecture has been studied extensively, and while evidence has been gathered in a number of examples including abelian varieties [15, 27, 22], it has so far only been proved for elliptic curves [33], the quartic K3 surface [36], and their products [7].

Kontsevich was also the first to suggest that homological mirror symmetry can be extended to a much more general setting [26], by considering *Landau-Ginzburg models*. Mathematically, a Landau-Ginzburg model is a pair (X, W) consisting of a variety X and a holomorphic function $W : X \to \mathbb{C}$ called *superpotential*. As far as homological mirror symmetry is concerned, the symplectic geometry of a Landau-Ginzburg model is determined by its Fukaya category, studied extensively by Seidel (see in particular [38]), while the B-model is determined by the triangulated category of singularities of the superpotential [31].

After the seminal works of Batyrev, Givental, Hori, Vafa, and many others, there are many known examples of Landau-Ginzburg mirrors to Fano varieties, especially in the toric case [13, 16, 12] where the examples can be understood using T-duality, generalising the ideas of Strominger, Yau, and Zaslow [42] beyond the case of Calabi-Yau manifolds. One direction of the mirror symmetry conjecture, in which the B-model consists of coherent sheaves on a Fano variety, has been established for toric Fano varieties in [10, 8, 44, 1, 14],

2010 *Mathematics Subject Classification.* Primary 53D37, 14J33; Secondary 53D40, 53D12, 18E30, 14F05.

as well as for for del Pezzo surfaces [9]. A proof in the other direction, in which the B-model is the category of matrix factorizations of the superpotential, has also been announced [5].

While Kontsevich's suggestion was originally studied for Fano manifolds, a more recent (and perhaps unexpected) development first proposed by the fourth author is that mirror symmetry also extends to varieties of general type, many of which also admit mirror Landau-Ginzburg models [21, 4, 19]. The first instance of homological mirror symmetry in this setting was established for the genus 2 curve by Seidel [39]. Namely, Seidel has shown that the derived Fukaya category of a smooth genus 2 curve is equivalent to the triangulated category of singularities of a certain 3-dimensional Landau-Ginzburg model (one notable feature of mirrors of varieties of general type is that they tend to be higher-dimensional). Seidel's argument was subsequently extended to higher genus curves [11], and to pairs of pants and their higher-dimensional analogues [41].

Unfortunately, the ordinary Fukaya category consisting of closed Lagrangians is insufficient in order to fully state the Homological mirror conjecture when the B-side is a Landau-Ginzburg model which fails to be proper or a variety which fails to be smooth. The structure sheaf of a non-proper component of the critical fiber of a Landau-Ginzburg model, or that of a singular point in the absence of any superpotential, generally have endomorphism algebras which are not of finite cohomological dimension, and hence cannot have mirrors in the ordinary Fukaya category, which is cohomologically finite. As all smooth affine varieties of the same dimension have isomorphic derived categories of coherent sheaves with compact support, one is led to seek a category of Lagrangians which would contain objects that are mirror to more general sheaves or matrix factorizations.

It is precisely to fill this role that the *wrapped Fukaya category* was constructed [6]. This Fukaya category, whose objects also include non-compact Lagrangian submanifolds, more accurately reflects the symplectic geometry of open symplectic manifolds, and by recent work [2, 18], is known in some generality to be homologically smooth in the sense of Kontsevich [28] (homological smoothness also holds for categories of matrix factorizations [30, 34, 29]).

In this paper, we give the first non-trivial verification that these categories are indeed relevant to Homological mirror symmetry: the non-compact Lagrangians we shall study will correspond to structure sheaves of irreducible components of a quasi-projective variety, considered as objects of its category of singularities. In particular, we provide the first computation of wrapped Fukaya categories beyond the case of cotangent bundles, studied in [3] using string topology.

As a final remark, we note that these categories should be of interest even when considering mirrors of compact symplectic manifolds. Indeed, since Seidel's ICM address [37], the standard approach to proving Homological mirror symmetry in this case is to first prove it for the complement of a divisor, then solve a deformation problem. As we have just explained, a proper formulation of Homological mirror symmetry for the complement involves the wrapped Fukaya category. More speculatively [40], one expects that the study of the wrapped Fukaya category will be amenable to sheaf-theoretic techniques. The starting point of such a program is the availability of natural restriction functors (to open subdomains) [6], which are expected to be mirror to restriction functors from the category of sheaves of a reducible variety to the category of sheaves on each component. This suggests that it might be possible to study homological mirror symmetry by a combination of sheaf-theoretic techniques and deformation theory, reducing the problem to elementary building blocks such as pairs of pants. While this remains a distant perspective, it very much motivates the present study.

1.2. **Main results.** In this paper, we study homological mirror symmetry for an open genus 0 curve C, namely, $\mathbb{P}^1$ minus a set of $n \geq 3$ points. A Landau-Ginzburg model mirror to C can be constructed by viewing C as a hypersurface in $(\mathbb{C}^*)^2$ (which can be compactified to a rational curve in $\mathbb{P}^1 \times \mathbb{P}^1$ or a Hirzebruch surface). The procedure described in [21] (or those in [19] or [4]) then yields a (noncompact) toric 3-fold $X(n)$, together with a superpotential $W : X(n) \to \mathbb{C}$, which we take as the mirror to C. For $n = 3$ the Landau-Ginzburg model $(X(3), W)$ is the three-dimensional affine space $\mathbb{C}^3$ with the superpotential $W = xyz$, while for $n > 3$ points $X(n)$ is more complicated (it is a toric resolution of a 3-dimensional singular affine toric variety); see Section 5 and Fig. 5 for details.

We focus on one side of homological mirror symmetry, in which we consider the wrapped Fukaya category of C (as defined in [6, 2]), and the associated triangulated derived category $D\mathcal{W}(C)$ (see Section (3j) of [38]). Our main theorem asserts that this triangulated category is equivalent to the triangulated category of singularities [31] of the singular fiber $W^{-1}(0)$ of $(X(n), W)$. In fact, we obtain a slightly stronger result than stated below, namely a quasi-equivalence between the natural A_∞-enhancements of these two categories.

Theorem 1.1. *Let C be the complement of a finite set of $n \geq 3$ points in $\mathbb{P}^1$, and let $(X(n), W)$ be the Landau-Ginzburg model defined in Section 5. Then the derived wrapped Fukaya category of C, $D\mathcal{W}(C)$, is equivalent to the triangulated category of singularities $D_{sg}(W^{-1}(0))$.*

The other side of homological mirror symmetry is generally considered to be out of reach of current technology for these examples, due to the singular nature of the critical locus of W.

Remark 1.2. The case $n = 0$ falls under the rubric of mirror symmetry for Fano varieties, and is easy to prove since the equatorial circle in S^2 is the unique non-displaceable

Lagrangian, and the mirror superpotential has exactly one non-degenerate isolated singularity. Mirror symmetry for $\mathbb{C}$ is trivial in this direction since the Fukaya category completely vanishes in this case, and the mirror superpotential has no critical point. Finally, the case $n = 2$ can be recovered as a degenerate case of our analysis, but was already essentially known to experts because the cylinder is symplectomorphic to the cotangent bundle of the circle, and Fukaya categories of cotangent bundles admit quite explicit descriptions using string topology [17, 3].

The general strategy of proof is similar to that used by Seidel for the genus 2 curve, and inspired by it. Namely, we identify specific generators of the respective categories (in Section 4 for $D\mathcal{W}(C)$, using a generation result proved in Appendix A, and in Section 6 for $D_{sg}(W^{-1}(0))$, and show that the corresponding A_∞ subcategories on either side are equivalent by appealing to an algebraic classification lemma (Section 3); see also Remark 4.2 for more about generation). A general result due to Keller (see Theorem 3.8 of [24] or Lemma 3.34 of [38]) implies that the categories $D\mathcal{W}(C)$ and $D_{sg}(W^{-1}(0))$ are therefore equivalent to the derived categories of the same A_∞ category, hence are equivalent to each other.

This strategy of proof can be extended to higher genus punctured Riemann surfaces, the main difference being that one needs to consider larger sets of generating objects (which in the general case leads to a slightly more technically involved argument). However, there is a special case in which the generalization of our result is particularly straightforward, namely the case of unramified cyclic covers of punctured spheres. The idea that Fukaya categories of unramified covers are closely related to those of the base is already present in Seidel's work [39] and the argument we use is again very similar (this approach can be used in higher dimensions as well, as evidenced in Sheridan's work [41]). As an illustration, we prove the following result in Section 7:

Theorem 1.3. *Given an unramified cyclic D-fold cover C of $\mathbb{P}^1 - \{3 \text{ points}\}$, there exists an action of $G = \mathbb{Z}/D$ on the Landau-Ginzburg model $(X(3), W)$ such that the derived wrapped Fukaya category $D\mathcal{W}(C)$ is equivalent to the equivariant triangulated category of singularities $D^G_{sg}(W^{-1}(0))$.*

Remark 1.4. The main difference between our approach and that developed in Seidel and Sheridan's papers [39, 41] is that, rather than compact (possibly immersed) Lagrangians, we consider the wrapped Fukaya category, which is strictly larger. The Floer homology of the immersed closed curve considered by Seidel in [39] can be recovered from our calculations, but not vice-versa. There is an obvious motivation for restricting to that particular object (and its higher dimensional analogue [41]): even though it does not determine the entire

A-model in the open case, it gives access to the Fukaya category of closed Riemann surfaces or projective Fermat hypersurfaces in a fairly direct manner. On the other hand, open Riemann surfaces and other exact symplectic manifolds are interesting both in themselves and as building blocks of more complicated manifolds.

We end this introduction with a brief outline of this paper's organization: Section 2 explicitly defines a category A, and introduces rudiments of Homological Algebra which are used, in the subsequent section, to classify A_∞ structures on this category up to equivalence. Section 4 proves that A is equivalent to a cohomological subcategory of the wrapped Fukaya category of a punctured sphere, and uses the classification result to identify the A_∞ structure induced by the count of homolorphic curves. In this Section, we also prove that our distinguished collection of objects strongly generate the wrapped Fukaya category.

The mirror superpotential is described in Section 5, and a collection of sheaves whose endomorphism algebra in the category of matrix factorizations is isomorphic to A is identified in the next Section, in which the A_∞ structure coming from the natural dg enhancement is also computed and a generation statement proved. At this stage, all the results needed for the proof of Theorem 1.1 are in place. Section 7 completes the main part of the paper by constructing the various categories appearing in the statement of Theorem 1.3. The paper ends with two appendices; the first proves a general result providing strict generators for wrapped Fukaya categories of curves, and the second shows that the categories of singularities that we study are idempotent complete.

Acknowledgements. M. A. was supported by a Clay Research Fellowship. D. A. was partially supported by NSF grants DMS-0652630 and DMS-1007177. A. E. was partially supported by the Dynasty Foundation, NSh grant 4713.2010.1, and by AG Laboratory HSE, RF government grant, ag. 11.G34.31.0023. L. K. was funded by NSF grant DMS-0600800, NSF FRG grant DMS-0652633, FWF grant P20778 and an ERC grant – GEMIS. D. O. was partially supported by RFBR grants 10-01-93113, 11-01-00336, 11-01-00568, NSh grant 4713.2010.1, and by AG Laboratory HSE, RF government grant, ag. 11.G34.31.0023. D. O. thanks the Simons Center for Geometry and Physics for its hospitality and stimulating atmosphere.

The authors acknowledge the hospitality of ESI (Vienna) and MSRI (Berkeley), where some of this work was carried out. We would also like to thank Paul Seidel for helpful discussions.

2. A_∞-structures

Let $\mathcal{A}$ be a small $\mathbb{Z}$-graded category over a field k, i.e. the morphism spaces $\mathcal{A}(X,Y)$ are $\mathbb{Z}$-graded k-modules and the compositions

$$\mathcal{A}(Y,Z)\otimes\mathcal{A}(X,Y)\longrightarrow\mathcal{A}(X,Z)$$

are morphisms of $\mathbb{Z}$-graded k-modules. By grading we will always mean $\mathbb{Z}$-gradings.

By an A_∞-structure on $\mathcal{A}$ we mean a collection of graded maps

$$m_k:\mathcal{A}(X_{k-1},X_k)\otimes\cdots\otimes\mathcal{A}(X_0,X_1)\longrightarrow\mathcal{A}(X_0,X_k),\quad X_i\in\mathcal{A},\quad k\geq 1$$

of degree $\deg(m_k)=2-k$, with $m_1=0$, and m_2 equal to the usual composition in $\mathcal{A}$, such that all together they define an A_∞-category, i.e. they satisfy the A_∞-associativity equations

$$\sum_{\substack{s,l,t\\ s+l+t=k}}(-1)^{s+lt}m_{k-l+1}(\mathrm{id}^{\otimes s}\otimes m_l\otimes\mathrm{id}^{\otimes t})=0, \tag{2.1}$$

for all $k\geq 1$. Note that additional signs appear when these formulas are applied to elements, according to the Koszul sign rule $(f\otimes g)(x\otimes y)=(-1)^{\deg g\cdot\deg x}f(x)\otimes g(y)$ (see [23, 38]).

Two A_∞-structures m and m' on $\mathcal{A}$ are said to be *strictly homotopic* if there exists an A_∞-functor f from $(\mathcal{A},m)$ to $(\mathcal{A},m')$ that acts identically on objects and for which $f_1=\mathrm{id}$ as well.

We also recall that an A_∞-functor f consists of a map $\bar f:\mathrm{Ob}(\mathcal{A},m)\to\mathrm{Ob}(\mathcal{A},m')$ and graded maps

$$f_k:\mathcal{A}(X_{k-1},X_k)\otimes\cdots\otimes\mathcal{A}(X_0,X_1)\longrightarrow\mathcal{A}(\bar fX_0,\bar fX_k),\quad X_i\in\mathcal{A},\quad k\geq 1$$

of degree $1-k$ which satisfy the equations

$$\sum_r\sum_{\substack{u_1,\dots,u_r\\ u_1+\cdots+u_r=k}}(-1)^\varepsilon m'_r(f_{u_1}\otimes\cdots\otimes f_{u_r})=\sum_{\substack{s,l,t,\\ s+l+t=k}}(-1)^{s+lt}f_{k-l+1}(\mathrm{id}^{\otimes s}\otimes m_l\otimes\mathrm{id}^{\otimes t}), \tag{2.2}$$

where the sign on the left hand side is given by $\varepsilon=(r-1)(u_1-1)+(r-2)(u_2-1)+\cdots+i_{r-1}$.

Now we introduce a k-linear category A that plays a central role in our considerations. It depends on an integer $n\geq 3$ and is defined by the following rule:

$$\mathrm{Ob}(A)=\{X_1,\dots,X_n\},\quad A(X_i,X_j)=\begin{cases}k[x_i,y_i]/(x_iy_i) & \text{for } j=i,\\ k[x_{i+1}]\,u_{i,i+1}=u_{i,i+1}\,k[y_i] & \text{for } j=i+1,\\ k[y_{i-1}]\,v_{i,i-1}=v_{i,i-1}\,k[x_i] & \text{for } j=i-1,\\ 0 & \text{otherwise.}\end{cases} \tag{2.3}$$

Here the indices are mod n, i.e. we put $X_{n+1}=X_1$, and $x_{n+1}=x_1$, $y_{n+1}=y_1$.

Compositions in this category are defined as follows. First of all, the above formulas already define $A(X_i, X_i)$ as k-algebras, and $A(X_i, X_j)$ as $A(X_i, X_i)$–$A(X_j, X_j)$-bimodules. To complete the definition, we set

$$(x_i^k u_{i-1,i}) \circ (v_{i,i-1} x_i^l) := x_i^{k+l+1}, \quad (v_{i,i-1} x_i^l) \circ (x_i^k u_{i-1,i}) := y_{i-1}^{k+l+1}$$

for any two morphisms $x_i^k u_{i-1,i} \in A(X_{i-1}, X_i)$ and $v_{i,i-1} x_i^l \in A(X_i, X_{i-1})$. All the other compositions vanish. Thus, A is defined as a k-linear category.

Choosing some collection of odd integers $p_1, \ldots, p_n, q_1, \ldots, q_n$ we can define a grading on A by the formulas

$$\deg(u_{i-1,i} : X_{i-1} \longrightarrow X_i) := p_i, \quad \deg(v_{i,i-1} : X_i \longrightarrow X_{i-1}) := q_i.$$

That implies $\deg x_i = \deg y_{i-1} = p_i + q_i$. All these gradings are refinements of the same $\mathbb{Z}/2$-grading on A.

In what follows we will require that the following conditions hold:

$$p_1, \ldots, p_n, q_1, \ldots, q_n \quad \text{are odd, and} \quad p_1 + \cdots + p_n = q_1 + \cdots + q_n = n - 2. \tag{2.4}$$

Definition 2.1. *For such collections of $p = \{p_i\}$ and $q = \{q_i\}$ we denote by $A_{(p,q)}$ the corresponding $\mathbb{Z}$-graded category.*

We are interested in describing all A_∞-structures on the category $A_{(p,q)}$. As we will see, these structures are in bijection with pairs (a, b) of elements $a, b \in k$.

Let $\mathcal{A}$ be a small $\mathbb{Z}$-graded category over a field k. It will be convenient to consider the bigraded Hochschild complex $CC^\bullet(\mathcal{A})^\bullet$,

$$CC^{k+l}(\mathcal{A})^l = \prod_{X_0, \ldots, X_k \in \mathcal{A}} \operatorname{Hom}^l(\mathcal{A}(X_{k-1}, X_k) \otimes \cdots \otimes \mathcal{A}(X_0, X_1),\ \mathcal{A}(X_0, X_k)).$$

with the Hochschild differential d of bidegree $(1, 0)$ defined by

$$dT(a_{k+1}, \ldots, a_1) = (-1)^{(k+l)(\deg(a_1)-1)+1} T(a_{k+1}, \ldots, a_2) a_1 +$$
$$+ \sum_{j=1}^{k} (-1)^{\epsilon_j + (k+l) - 1} T(a_{k+1}, \ldots, a_{j+1} a_j, \ldots, a_1) + (-1)^{\epsilon_k + (k+l)} a_{k+1} T(a_k, \ldots, a_1),$$

where the sign is defined by the rule $\epsilon_j = \sum_{i=1}^{j} \deg a_i - j$. We denote by $HH^{k+l}(\mathcal{A})^l$ the bigraded Hochschild cohomology.

Denote by $A_\infty S(\mathcal{A})$ the set of A_∞-structures on $\mathcal{A}$ up to strict homotopy.

Basic obstruction theory implies the following proposition, which will be sufficient for our purposes.

Proposition 2.2. *Assume that the small $\mathbb{Z}$-graded k-linear category $\mathcal{A}$ satisfies the following conditions*

$$HH^2(\mathcal{A})^j = 0 \text{ for } j \leq -1 \text{ and } j \neq -l, \tag{2.5}$$

and

$$HH^3(\mathcal{A})^j = 0 \text{ for } j < -l, \tag{2.6}$$

for some positive integer $l \geq 1$. Then for any $\phi \in HH^2(\mathcal{A})^{-l}$ there is an A_∞-structure m^ϕ with $m_3 = \cdots = m_{l+1} = 0$, for which the class of m_{l+2} in $HH^2(\mathcal{A})^{-l}$ is equal to ϕ. Moreover, the natural map

$$HH^2(\mathcal{A})^{-l} \to A_\infty S(\mathcal{A}), \quad \phi \mapsto m_\phi,$$

is a surjection, i.e. any other A_∞-structure is strictly homotopic to m_ϕ.

To prove this proposition, we recall some well-known statements from obstruction theory. Let m be an A_∞-structure on a graded category $\mathcal{A}$. Let us consider the A_∞-constraint (2.1) of order $k+1$. Since $m_1 = 0$ it is the first constraint that involves m_k. Moreover, it can be written in the form

$$dm_k = \Phi_k(m_3, \ldots, m_{k-1}), \tag{2.7}$$

where d is the Hochschild differential and $\Phi_k = \Phi_k(m_3, \ldots, m_{k-1})$ is a quadratic expression.

Similarly, let m and m' be two A_∞-structures on a graded category $\mathcal{A}$, and let $f = (\bar{f} = \mathrm{id}; f_1 = \mathrm{id}, f_2, f_3, \ldots)$ be a strict homotopy between m and m'. Since $m_1 = m'_1 = 0$ the order $k+1$ A_∞-constraint (2.2) is the first one that contains f_k. It can be written as

$$\begin{aligned} df_k = \Psi_k(f_2, \ldots, f_{k-1}; m_3, \ldots, m_{k+1}; m'_3, \ldots, m'_{k+1}) = \\ = \Psi'_k(f_2, \ldots, f_{k-1}; m_3, \ldots, m_k; m'_3, \ldots, m'_k) + m'_{k+1} - m_{k+1} \end{aligned} \tag{2.8}$$

where d is the Hochschild differential and Ψ_k is a polynomial expression. The following lemma is well-known and can be proved by a direct calculation.

Lemma 2.3. *In the above notations, let d be the Hochschild differential.*

(1) *Assume that the first k A_∞-constraints (2.1), which depend only on $m_{<k}$, hold. Then*
$$d\Phi_k(m_3, \ldots, m_{k-1}) = 0.$$
(2) *Let m and m' be two A_∞-structures on a graded category $\mathcal{A}$, and f a strict homotopy between them. Assume that the first k A_∞-constraints (2.2), which depend only on $f_{<k}$, hold. Then*
$$d\Psi_k(f_2, \ldots, f_{k-1}; m, m') = 0.$$

The following lemma is a direct consequence of the k^{th} A_∞-constraint (2.2).

Lemma 2.4. *Let m and m' be two A_∞-structures on a graded category $\mathcal{A}$. Let $f : (\mathcal{A}, m) \to (\mathcal{A}, m')$ be an A_∞-homomorphism with $f_1 = \mathrm{id}$, and $f_i = 0$ for $1 < i < k-1$. Then $m_i = m'_i$ for $i < k$ and $df_{k-1} = m'_k - m_k$.*

Proof of Proposition 2.2. We define the desired surjection as follows. Let $\phi \in HH^2(\mathcal{A})^{-l}$ be some class, and $\widetilde{\phi} \in CC^2(\mathcal{A})^{-l}$ its representative. Consider the partial A_∞-structure $(m_3, \dots, m_{l+2})$ with

$$m_{l+2} = \widetilde{\phi}, \quad m_3 = \dots = m_{l+1} = 0.$$

The maps $m_{\le l+2}$ satisfy all the required equations (2.1) which do not involve $m_{>l+2}$ (there is only one nontrivial such equation, $dm_{l+2} = 0$). By induction on k, the equation

$$dm_k = \Phi_k(m_3, \dots, m_{k-1})$$

has a solution for each $k > l+2$, since we know from part (1) of Lemma 2.3 that $d\Phi_k = 0$ and from condition (2.6) that $HH^3(\mathcal{A})^j = 0$ when $j < -l$. This means that $(m_3, \dots, m_{l+2})$ lifts to some A_∞-structure $m^{\widetilde{\phi}}$ on $\mathcal{A}$.

Moreover, by condition (2.5) we have $HH^2(\mathcal{A})^j = 0$ when $j < -l$, and by Lemma 2.3 (2) we know that $d\Psi_k = 0$. This implies that the equation (2.8) can be solved for all $k > l+1$, i.e. the lift is unique up to strict homotopy. Finally, similar considerations and Lemma 2.4 give that the resulting element $m^{\widetilde{\phi}} \in A_\infty S(\mathcal{A})$ depends only on ϕ, not on $\widetilde{\phi}$.

Therefore, the map $HH^2(\mathcal{A})^{-l} \to A_\infty S(\mathcal{A})$ is well-defined. Now we show that it is surjective. Let us consider an A_∞-structure m' on $\mathcal{A}$ and let us take some A_∞-structure $m^{\widetilde{\phi}}$ with $m_3 = \dots = m_{l+1} = 0$ and $m_{l+2} = \widetilde{\phi}$ as above. By condition (2.5) $HH^2(\mathcal{A})^j = 0$ for all $j \le -1$ and $j \ne l$. Hence by (2) of Lemma 2.3 we can construct a strict homotopy f between m' and $m^{\widetilde{\phi}}$ if and only if the expression Ψ_{l+1} from (2.8) is exact. Since Ψ_{l+1} depends linearly on m_{l+2}, we can find $\widetilde{\phi}$ such that the class of Ψ_{l+1} in the cohomology group $HH^2(\mathcal{A})^{-l}$ vanishes; hence, for this choice of $\widetilde{\phi}$, the A_∞-structure m' will be strictly homotopic to $m^{\widetilde{\phi}}$. This completes the proof of the proposition. □

3. A classification of A_∞-structures

In this section we describe all A_∞-structures on the category $A_{(p,q)}$. The main technical result of this section is the following proposition:

Proposition 3.1. *Let A be the category with $n \ge 3$ objects defined by* (2.3). *Then*

(1) *For any two elements $a, b \in k$, there exists a $\mathbb{Z}/2$-graded A_∞-structure $m^{a,b}$ on A, compatible with all $\mathbb{Z}$-gradings satisfying* (2.4), *such that $m_3^{a,b} = \dots = m_{n-1}^{a,b} = 0$*

and

$$m_n^{a,b}(u_{i-1,i}, u_{i-2,i-1}, \ldots, u_{i,i+1})(0) = a, \quad m_n^{a,b}(v_{i+1,i}, v_{i+2,i+1}, \ldots, v_{i,i-1})(0) = b$$

for any $1 \le i \le n$, *where* $\cdot(0)$ *means the constant coefficient of an element of* $A(X_i, X_i)$, *i.e. the coefficient of* id_{X_i}.

(2) *Moreover, for any* $\mathbb{Z}$*-grading* $A_{(p,q)}$ *where* (p,q) *satisfy* (2.4), *the map*

$$k^2 \to A_\infty S(A_{(p,q)}), \quad (a,b) \mapsto m^{a,b},$$

is a bijection, i.e. any A_∞*-structure* m *on* $A_{(p,q)}$ *is strictly homotopic to* $m^{a,b}$ *with*

$$a = m_n(u_{n,1}, u_{n-1,n}, \ldots, u_{1,2})(0), \quad b = m_n(v_{2,1}, v_{3,2}, \ldots, v_{1,n})(0). \tag{3.1}$$

The proof of this proposition essentially reduces to the computation of the Hochschild cohomology of $A_{(p,q)}$.

Lemma 3.2. *Let* $A_{(p,q)}$ *be the* $\mathbb{Z}$*-graded category with* $n \ge 3$ *objects as in Definition 2.1. Then the bigraded Hochschild cohomology of* $A_{(p,q)}$ *is the following:*

$$HH^d(A_{(p,q)})^j \cong \begin{cases} k^2 & \text{for each } d \ge 2 \quad \text{when} \quad j = \left\lfloor \dfrac{d}{2} \right\rfloor (2-n), \\ 0 & \text{in all other cases when} \quad d - j \ge 2. \end{cases}$$

Proof. We have a subcomplex

$$CC^\bullet_{red}(\mathcal{A})^\bullet \subset CC^\bullet(\mathcal{A})^\bullet, \tag{3.2}$$

the so-called reduced Hochschild complex, which consists of cochains that vanish on any sequence of morphisms containing some identity morphism. It is classically known that the inclusion (3.2) is a quasi-isomorphism. We will compute Hochschild cohomology using the reduced Hochschild complex. For convenience, we will write just A instead of $A_{(p,q)}$. Let

$$\widetilde{A} = \bigoplus_{i,j} A(X_i, X_j).$$

This is a graded algebra. We have a non-unital graded algebra

$$A_{red} := \ker\Big(\bigoplus_{i,j} A(X_i, X_j) \to \bigoplus_i k \cdot \mathrm{id}_{X_i}\Big).$$

Let $R = \bigoplus_i k \cdot \mathrm{id}_{X_i}$. Then both A_{red} and $\widetilde{A}$ are R-R-bimodules, and

$$CC^{k+l}_{red}(A)^l = \mathrm{Hom}^l_{R-R}(A_{red}{}^{\otimes_R k}, \widetilde{A}), \quad k \ge 0.$$

Denote by $A_i \subset A_{red}$ the subalgebra generated by $u_{i-1,i}$ and $v_{i,i-1}$. Then we have an isomorphism

$$A_{red} \cong \bigoplus_i A_i$$

of non-unital graded algebras (because $A_i \cdot A_j = 0$ for $i \neq j$).

Consider the bar complex of R–R-bimodules

$$K_i^\bullet = \overline{T}(sA_i) = \bigoplus_{m>0} (sA_i)^{\otimes_R m},$$

where $(sA_i)^p = (A_i)^{p+1}$ and the differential is the bar differential

$$D(sa_k \otimes \cdots \otimes sa_1) = \sum_{i=1}^{k-1} (-1)^{\epsilon_i} sa_k \otimes \cdots \otimes sa_{i+1}a_i \otimes \cdots sa_1$$

with $\epsilon_i = \sum_{j \leq i} \deg sa_j$.

Denote by $A_i(d) \subset A_i$, $d > 0$, the 2-dimensional subspace generated by the two products of $u_{i-1,i}$ and $v_{i,i-1}$ of length d, i.e. $A_i(2m+1)$ is generated by $x_i^m u_{i-1,i}$ and $v_{i,i-1}x_i^m$ while $A(2m)$ is generated by x_i^m and y_{i-1}^m. Consider the subcomplex

$$K_i^\bullet(d) \subset K_i^\bullet, \quad K_i^\bullet(d) = \bigoplus_{\substack{d_1+\cdots+d_l=d,\\ l>0}} sA_i(d_1) \otimes_R sA_i(d_2) \otimes_R \cdots \otimes_R sA_i(d_l).$$

Lemma 3.3. $K_i(1) \cong sA_i(1)$, *and for* $d > 1$ *the complex* $K_i^\bullet(d)$ *is acyclic.*

Proof. The result is obvious for $d = 1$. For $d \geq 2$, we subdivide the complex $K_i(d)$ into two parts, according to whether $d_l = 1$ or $d_l > 1$. The first part is $K_i(d-1) \otimes_R sA_i(1)$. We also note that the product map $A_i(d_l - 1) \otimes_R A_i(1) \to A_i(d_l)$ is an isomorphism. Hence the second part of the complex is isomorphic to $K_i(d-1) \otimes_R A_i(1)$. Using these identifications, we conclude that $K_i(d)$ is isomorphic to the total complex of the bicomplex $K_i(d-1) \otimes_R A_i(1) \to K_i(d-1) \otimes_R A_i(1)$, where the connecting map is the identity map. It is therefore acyclic. □

Now let

$$K^\bullet = T(sA_{red}) = \bigoplus_{m \geq 0} (sA_{red})^{\otimes_R m}.$$

We have an isomorphism of complexes

$$K^\bullet = R \oplus \bigoplus_{\substack{s>0\\ i_t \neq i_{t+1}}} K_{i_1}^\bullet \otimes_R \cdots \otimes_R K_{i_s}^\bullet,$$

because $A_i \cdot A_j = 0$ for $i \neq j$. Define subcomplexes

$$K^\bullet(0) = R, \quad K^\bullet(d) = \bigoplus_{\substack{s>0\\ d_1+\cdots+d_s=d,\\ i_t \neq i_{t+1}}} K_{i_1}^\bullet(d_1) \otimes_R \cdots \otimes_R K_{i_s}^\bullet(d_s) \ \text{ for } \ d \geq 1.$$

Consider the full decreasing filtration

$$CC^\bullet(A)_{red}^\bullet = L_0^\bullet(A)^\bullet \supset L_1^\bullet(A)^\bullet \supset L_2^\bullet(A)^\bullet \supset \ldots,$$

where $L^\bullet_r(A)^\bullet$ consists of all cochains vanishing on $K^\bullet(i)$ for $0 \le i < r$.

Denote by $\mathrm{Gr}^\bullet_r(A)^\bullet = L^\bullet_r(A)^\bullet / L^\bullet_{r+1}(A)^\bullet$ the associated graded factors of this filtration. The Hochschild differential d induces a differential

$$d_0 : \mathrm{Gr}^\bullet_r(A)^\bullet \to \mathrm{Gr}^{\bullet+1}_r(A)^\bullet.$$

It is easy to see that d_0 coincides with a differential defined by the bar differential D on $K^\bullet$. Therefore, Lemma 3.3 implies that for $r \ge 1$ we have

$$H^{r+j}(\mathrm{Gr}^\bullet_r(A)^\bullet)^j = \\ = \mathrm{Hom}^j_{R-R}\Big(\bigoplus_{t\in\mathbb{Z}/n\mathbb{Z}} k\cdot u_{t-1,t}\otimes\cdots\otimes u_{t-r,t-r+1} \;\oplus \bigoplus_{t\in\mathbb{Z}/n\mathbb{Z}} k\cdot v_{t+1,t}\otimes\cdots\otimes v_{t+r,t+r-1},\ \widetilde{A}\Big),$$

and

$$H^{i+j}(\mathrm{Gr}^\bullet_r(A)^\bullet)^j = 0 \quad \text{for} \quad i \ne r.$$

The first differential

$$d_1 : H^{r+j}(\mathrm{Gr}^\bullet_r(A)^\bullet)^j \to H^{r+j+1}(\mathrm{Gr}^\bullet_r(A)^\bullet)^j$$

in the spectral sequence $E_1^{r,j} = H^{r+j}(\mathrm{Gr}^\bullet_r(A)^\bullet)^j$ is given by the formula

$$\begin{cases} d_1\phi(u_{t-1,t}, u_{t-2,t-1}, \ldots, u_{t-r,t-r+1}) = & \pm u_{t-1,t}\phi(u_{t-2,t-1}, \ldots, u_{t-r,t-r+1}) \\ & \pm\phi(u_{t-1,t}, \ldots, u_{t-r+1,t-r+2})u_{t-r,t-r+1}, \\ d_1\phi(v_{t+1,t}, v_{t+2,t+1}, \ldots, v_{t+r,t+r-1}) = & \pm v_{t+1,t}\phi(v_{t+2,t+1}, \ldots, v_{t+r,t+r-1}) \\ & \pm\phi(v_{t+1,t}, \ldots, v_{t+r-1,t+r-2})v_{t+r,t+r-1}. \end{cases}$$

It is clear that $H^{r+j}(\mathrm{Gr}^\bullet_r(A)^\bullet)^j \ne 0$ only for $r \equiv 0, \pm 1 \bmod n$ and the spectral sequence $(E_1^{\bullet,\bullet}, d_1)$ consists of the following simple complexes

$$0 \to H^{mn+j-1}(\mathrm{Gr}^\bullet_{mn-1}(A)^\bullet)^j \to H^{mn+j}(\mathrm{Gr}^\bullet_{mn}(A)^\bullet)^j \to \\ \to H^{mn+j+1}(\mathrm{Gr}^\bullet_{mn+1}(A)^\bullet)^j \to 0. \tag{3.3}$$

Let $m > 0$. Now, if $j \ne m(2-n)$ then the complexes (3.3) are acyclic. If $j = m(n-2)$, then the complex (3.3) has only two nontrivial terms and is

$$0 \to \mathrm{Hom}_{R-R}\Big(\bigoplus_{t\in\mathbb{Z}/n\mathbb{Z}} k\cdot u_{t-1,t}\otimes\cdots\otimes u_{t-mn,t-mn+1} \oplus \bigoplus_{t\in\mathbb{Z}/n\mathbb{Z}} k\cdot v_{t+1,t}\otimes\cdots\otimes v_{t+mn,t+mn-1}, R\Big) \\ \to \mathrm{Hom}_{R-R}\Big(\bigoplus_{t\in\mathbb{Z}/n\mathbb{Z}} k\cdot u_{t-1,t}\otimes\cdots\otimes u_{t-mn-1,t-mn} \oplus \bigoplus_{t\in\mathbb{Z}/n\mathbb{Z}} k\cdot v_{t+1,t}\otimes\cdots\otimes v_{t+mn+1,t+mn}, \\ \bigoplus_{t\in\mathbb{Z}/n\mathbb{Z}} k\cdot u_{t-1,t} \oplus \bigoplus_{t\in\mathbb{Z}/n\mathbb{Z}} k\cdot v_{t+1,t}\Big) \to 0.$$

Thus, the computation of the cohomology of d_1 reduces to an easy computation of the kernel and the cokernel of this map. For $m > 0$ we obtain that the cohomology of d_1 is the following:

$$\begin{aligned}
&H^{2m}(E_1^{\bullet,\bullet}, d_1)^{m(2-n)} \cong k^2, && \phi^{a,b}(u_{i-1,i}, u_{i-2,i-1}, \dots, u_{i,i+1}) = a \cdot \mathrm{id}_{X_i}, \\
& && \phi^{a,b}(v_{i+1,i}, v_{i+2,i+1}, \dots, v_{i,i-1}) = b \cdot \mathrm{id}_{X_i}, \quad a, b \in k, \\
&H^{2m+1}(E_1^{\bullet,\bullet}, d_1)^{m(2-n)} \cong k^2, && \psi^{c,d}(u_{i-1,i}, u_{i-2,i-1}, \dots, u_{i-1,i}) = \delta_{i1} \cdot c \cdot u_{i-1,i}, \\
& && \psi^{c,d}(v_{i+1,i}, v_{i+2,i+1}, \dots, v_{i+1,i}) = \delta_{i1} \cdot d \cdot v_{i+1,i}, \;\; c, d \in k, \\
&H^{i+j}(E_1^{\bullet,\bullet}, d_1)^j = 0 && \text{in all other cases with } i \geq 2.
\end{aligned}$$

It is easy to see that the spectral sequence degenerates at the $E_2^{\bullet,\bullet}$ term, i.e. all these classes can be lifted to actual Hochschild cohomology classes. This proves Lemma 3.2. □

Proof of Proposition 3.1. Part (1) directly follows from Lemma 3.2 and Proposition 2.2.

Lemma 3.2 and Proposition 2.2 also imply that the map $(a, b) \mapsto m^{a,b}$ is a surjection on $A_\infty S(A_{p,q})$. Further, it is straightforward to check that the coefficients (3.1) are invariant under strict homotopy. This proves part (2) of the proposition. □

Remark 3.4. Note that autoequivalences of the graded category $A_{p,q}$ act on the set of A_∞-structures $A_\infty S(A_{p,q})$. In particular, it is easy to see that all A_∞-structures $m^{a,b}$ with $a \neq 0$, $b \neq 0$ yield equivalent A_∞-categories, all of them quasi-equivalent to $m^{1,1}$. We also have three degenerate A_∞-categories defined by $m^{0,1}, m^{1,0}$ and $m^{0,0}$, where the last-mentioned coincides with the category $A_{p,q}$ itself.

4. The wrapped Fukaya category of C

In this section we study the wrapped Fukaya category of C. Recall that the wrapped Fukaya category of an exact symplectic manifold (equipped with a Liouville structure) is an A_∞-category whose objects are (graded) exact Lagrangian submanifolds which are invariant under the Liouville flow outside of a compact subset. Morphisms and compositions are defined by considering Lagrangian Floer intersection theory perturbed by the flow generated by a Hamiltonian function H which is quadratic at infinity. Specifically, the wrapped Floer complex $\mathrm{Hom}(L, L') = CW^*(L, L')$ is generated by time 1 trajectories of the Hamiltonian vector field X_H which connect L to L', or equivalently, by points in $\phi_H^1(L) \cap L'$; compositions count solutions to a perturbed Cauchy-Riemann equation. In the specific case of punctured spheres, these notions will be clarified over the course of the discussion; the reader is referred to [2, Sections 2–4] for a complete definition (see also [6] for a different construction).

The goal of this section is to prove the following:

Theorem 4.1. *The wrapped Fukaya category of C (the complement of $n \geq 3$ points in $\mathbb{P}^1$) is strictly generated by n objects $L_1, \dots, L_n$ such that*

$$\bigoplus_{i,j} \operatorname{Hom}(L_i, L_j) \simeq \bigoplus_{i,j} A(X_i, X_j),$$

where A is the category defined in (2.3) (*with any grading satisfying* (2.4)), *and the associated A_∞-structure is strictly homotopic to $m^{1,1}$.*

We now make a couple of remarks in order to clarify the meaning of this statement.

Remark 4.2. (1) A given set of objects is usually said to generate a triangulated category $\mathcal{T}$ when the smallest triangulated subcategory of $\mathcal{T}$ containing the given objects and closed under taking direct summands is the whole category $\mathcal{T}$; or equivalently, when every object of $\mathcal{T}$ is isomorphic to a direct summand of a complex built out of the given objects. In the symplectic geometry literature this concept is sometimes called "split-generation" (cf. e.g. [2]). By contrast, in this paper we always consider a stronger notion of generation, in which direct summands are not allowed: namely, we say that $\mathcal{T}$ is *strictly generated* by the given objects if the minimal triangulated subcategory containing these objects is $\mathcal{T}$.

(2) The A_∞-category $\mathcal{W}(C)$ is not triangulated, however it admits a natural triangulated enlargement, the A_∞-category of *twisted complexes* $\operatorname{Tw}\mathcal{W}(C)$ (see e.g. section 3 of [38]). The derived wrapped Fukaya category, appearing in the statement of Theorem 1.1, is then defined to be the homotopy category $D\mathcal{W}(C) = H^0(\operatorname{Tw}\mathcal{W}(C))$; this is an honest triangulated category. By definition, we say that $\mathcal{W}(C)$ is strictly generated by the objects $L_1, \dots, L_n$ if these objects strictly generate the derived category $D\mathcal{W}(C)$; or equivalently, if every object of $\mathcal{W}(C)$ is quasi-isomorphic in $\operatorname{Tw}\mathcal{W}(C)$ to a twisted complex built out of the objects $L_1, \dots, L_n$ and their shifts.

(3) For the examples we consider in this paper, it turns out that the difference between strict generation and split-generation is not important. Indeed, in Appendix B we show that the triangulated categories $D\mathcal{W}(C)$ and $D_{sg}(W^{-1}(0))$ are actually idempotent complete.

In order to construct the wrapped Fukaya category $\mathcal{W}(C)$, we equip C with a Liouville structure, i.e. a 1-form λ whose differential is a symplectic form $d\lambda = \omega$, and whose associated Liouville vector field Z (defined by $i_Z\omega = \lambda$) is outward pointing near the punctures; thus (C, λ) has n cylindrical ends modelled on $(S^1 \times [1, \infty), r\, d\theta)$. The objects of $\mathcal{W}(C)$ are (graded) exact Lagrangian submanifolds of C which are invariant under the Liouville flow (i.e., radial) inside each cylindrical end (see [6, 2] for details; we will use the same setup as in [2]). As a consequence of Theorem 4.1, the wrapped Fukaya category is independent of the choice of λ; this can be *a priori* verified using the fact that, up to adding the differential

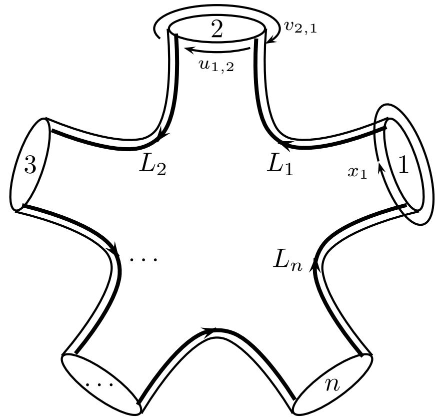

FIGURE 1. The generators of $\mathcal{W}(C)$

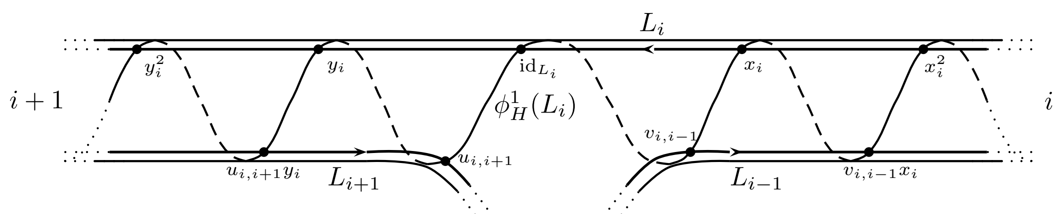

FIGURE 2. Generators of the wrapped Floer complexes

of a compactly supported function, any two Liouville structures can be intertwined by a symplectomorphism.

We specifically consider n disjoint oriented properly embedded arcs $L_1, \dots, L_n \subset C$, where L_i runs from the i^{th} to the $i+1^{\text{st}}$ cylindrical end of C (counting mod n as usual), as shown in Figure 1. To simplify some aspects of the discussion below, we will assume that $L_1, \dots, L_n$ are invariant under the Liouville flow everywhere (not just at infinity); this can be ensured e.g. by constructing the Liouville structure starting from two discs (the front and back of Figure 1) and attaching n handles whose co-cores are the L_i.

Recall that the wrapped Floer complex $CW^*(L_i, L_j)$ is generated by time 1 chords of the flow ϕ_H^t generated by a Hamiltonian $H : C \to \mathbb{R}$ which is quadratic at infinity (i.e., $H(r, \theta) = r^2$ in the cylindrical ends), or equivalently by (transverse) intersection points of $\phi_H^1(L_i) \cap L_j$. Without loss of generality we can assume that, for each $1 \le i \le n$, $H_{|L_i}$ is a Morse function with a unique minimum.

Lemma 4.3. *The Floer complex $CW^*(L_i, L_j)$ is naturally isomorphic to the vector space $A(X_i, X_j)$ defined by* (2.3)*. Moreover, for every choice of $\mathbb{Z}$-grading satisfying* (2.4) *there exists a choice of graded lifts of $L_1, \dots, L_n$ such that the isomorphism preserves gradings.*

Proof. The intersections between $\phi_H^1(L_i)$ and L_i (resp. $L_{i\pm 1}$) are pictured in Figure 2. The point of $\phi_H^1(L_i) \cap L_i$ which corresponds to the minimum of $H_{|L_i}$ is labeled by the identity element, while the successive intersections in the i^{th} end are labeled by powers of x_i, and similarly those in the $(i+1)^{\text{st}}$ end are labeled by powers of y_i. The generators of $CW^*(L_i, L_{i+1})$ (i.e., points of $\phi_H^1(L_i) \cap L_{i+1}$) are labeled by $u_{i,i+1}y_i^k$, $k = 0, 1, \ldots,$ and similarly the generators of $CW^*(L_i, L_{i-1})$ are labeled by $v_{i,i-1}x_i^k$ (see Figure 2).

Recall that a $\mathbb{Z}$-grading on Floer complexes requires the choice of a trivialization of TC. Denote by $d_i \in \mathbb{Z}$ the rotation number of a simple closed curve encircling the i^{th} puncture of C with respect to the chosen trivialization: by an Euler characteristic argument, $\sum d_i = n - 2$. Observing that each rotation around the i^{th} cylindrical end contributes $2d_i$ to the Maslov index, we obtain that $\deg(x_i^k) = 2kd_i$, and similarly $\deg(y_i^k) = 2kd_{i+1}$.

The freedom to choose graded lifts of the Lagrangians L_i (compatibly with the given orientations) means that $p_i = \deg(u_{i-1,i})$ can be any odd integer for $i = 2, \ldots, n$; however, considering the n-gon obtained by deforming the front half of Figure 1, we obtain the relation $p_1 + \cdots + p_n = n - 2$. Moreover, comparing the Maslov indices of the various morphisms between L_{i-1} and L_i in the i^{th} end we obtain that $\deg(x_i^k u_{i-1,i}) = p_i + 2kd_i$, $\deg(v_{i,i-1}) = 2d_i - p_i$, and $\deg(v_{i,i-1}x_i^l) = 2d_i - p_i + 2ld_i$. Setting $q_i = 2d_i - p_i$, this completes the proof. □

It follows immediately from Lemma 4.3 that the Floer differential on $CW^*(L_i, L_j)$ is identically zero, since the degrees of the generators all have the same parity.

Lemma 4.4. *There is a natural isomorphism of algebras*

$$\bigoplus_{i,j} HW^*(L_i, L_j) \simeq \bigoplus_{i,j} A(X_i, X_j)$$

where A is the k-linear category defined by (2.3)*.*

Proof. Recall from [2, Section 3.2] that the product on wrapped Floer cohomology can be defined by counting solutions to a perturbed Cauchy-Riemann equation. Namely, one considers finite energy maps $u : S \to C$ satisfying an equation of the form

$$(du - X_H \otimes \alpha)^{0,1} = 0. \tag{4.1}$$

Here the domain S is a disc with three strip-like ends, and u is required to map ∂S to the images of the respective Lagrangians under suitable Liouville rescalings (in our case L_i is invariant under the Liouville flow, so ∂S is mapped to L_i); X_H is the Hamiltonian vector field generated by H, and α is a closed 1-form on S such that $\alpha_{|\partial S} = 0$ and which is standard in the strip-like ends (modelled on dt for the input ends, $2\,dt$ for the output end).

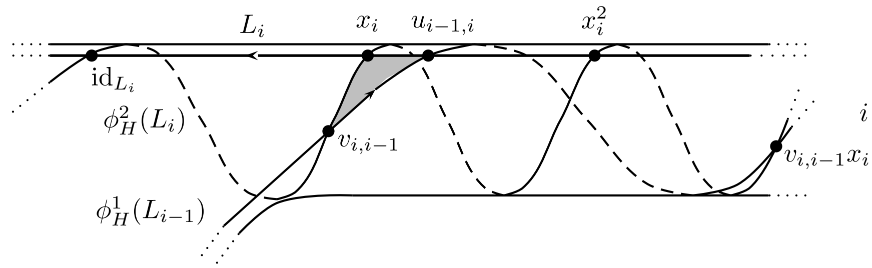

FIGURE 3. A holomorphic triangle contributing to the product

(Further perturbations of H and J would be required to achieve transversality in general, but are not necessary in our case.)

The equation (4.1) can be rewritten as a standard holomorphic curve equation (with a domain-dependent almost-complex structure) by considering

$$\tilde{u} = \phi_H^\tau \circ u : S \to C,$$

where $\tau : S \to [0,2]$ is a primitive of α. The product on $CW^*(L_j, L_k) \otimes CW^*(L_i, L_j)$ is then the usual Floer product

$$CF^*(\phi_H^1(L_j), L_k) \otimes CF^*(\phi_H^2(L_i), \phi_H^1(L_j)) \to CF^*(\phi_H^2(L_i), L_k),$$

where the right-hand side is identified with $CW^*(L_i, L_k)$ by a rescaling trick [2].

With this understood, since we are interested in rigid holomorphic discs, the computation of the product structure is simply a matter of identifying all immersed polygonal regions in C with boundaries on $\phi_H^2(L_i)$, $\phi_H^1(L_j)$ and L_k and satisfying a local convexity condition at the corners. (Simultaneous compatibility of the product structure with all $\mathbb{Z}$-gradings satisfying (2.4) drastically reduces the number of cases to consider.) Signs are determined as in [38, Section 13], and in our case they all turn out to be positive for parity reasons.

As an example, Figure 3 shows the triangle which yields the identity $u_{i-1,i} \circ v_{i,i-1} = x_i$. (The triangle corresponding to $u_{i-1,i} \circ (v_{i,i-1}x_i) = x_i^2$ is also visible.) □

Lemma 4.5. *In $\mathcal{W}(C)$ we have*

$$m_n(u_{i-1,i}, u_{i-2,i-1}, \dots, u_{i,i+1}) = \mathrm{id}_{L_i} \quad \text{and} \quad m_n(v_{i+1,i}, v_{i+2,i+1}, \dots, v_{i,i-1}) = (-1)^n \mathrm{id}_{L_i}.$$

Proof. Since $m_n(u_{i-1,i}, \dots, u_{i,i+1})$ has degree 0 for all gradings satisfying (2.4), it must be a scalar multiple of id_{L_i}. By the same argument as in Lemma 4.4, the calculation reduces to an enumeration of immersed $(n+1)$-sided polygonal regions with boundary on $\phi_H^n(L_i)$, $\phi_H^{n-1}(L_{i+1}), \dots, \phi_H^1(L_{i-1})$, and L_i, with locally convex corners at the prescribed intersection points. Recall that $u_{j,j+1}$ is the first intersection point between the images of L_j and L_{j+1} created by the wrapping flow inside the $(j+1)^{\text{st}}$ cylindrical end, and can also be visualized

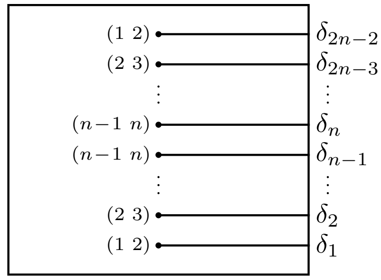

FIGURE 4. A simple branched cover $\pi : C \to \mathbb{C}$

as a chord from L_j to L_{j+1} as pictured in Figure 1. The only polygonal region which contributes to m_n is therefore the front half of Figure 1 (deformed by the wrapping flow). Since the orientation of the boundary of the polygon agrees with that of the L_j's, its contribution to the coefficient of id_{L_i} in $m_n(u_{i-1,i}, u_{i-2,i-1}, \dots, u_{i,i+1})$ is $+1$ (cf. [38, §13]).

The argument is the same for $m_n(v_{i+1,i}, \dots, v_{i,i-1})$, except the polygon which contributes now corresponds to the back half of Figure 1. Since the orientation of the boundary of the polygon differs from that of the L_j's, and $\deg(v_{j,j-1}) = q_j$ is odd for all $j = 1, \dots, n$, the coefficient of id_{L_i} is now $(-1)^n$. □

By Lemma 3.1, we conclude that the A_∞-structure on $\bigoplus_{i,j} \mathrm{Hom}(L_i, L_j)$ is strictly homotopic to $m^{1,(-1)^n}$. The sign discrepancy can be corrected by changing the identification between the two categories: namely, the automorphism of $\widetilde{A}$ which maps $u_{i,i+1}$ to itself, $v_{i,i-1}$ to $-v_{i,i-1}$, and x_i to $-x_i$ intertwines the A_∞-structures $m^{1,(-1)^n}$ and $m^{1,1}$.

The final ingredient needed for Theorem 4.1 is the following generation statement:

Lemma 4.6. *$\mathcal{W}(C)$ is strictly generated by $L_1, \dots, L_{n-1}$.*

Proof. Observe that C can be viewed as an n-fold simple branched covering of $\mathbb{C}$ with $2n-2$ branch points, around which the monodromies are successively (1 2), (2 3), …, $(n-1\ n)$, $(n-1\ n)$, …, (2 3), (1 2); see Figure 4. (Since the product of these transpositions is the identity, the monodromy at infinity is trivial, and it is easy to check that the n-fold cover we have described is indeed an n-punctured $\mathbb{P}^1$).

The $2n-2$ thimbles $\delta_1, \dots, \delta_{2n-2}$ are disjoint properly embedded arcs in C, projecting to the arcs shown in Figure 4. We claim that they are respectively isotopic to $L_1, \dots, L_{n-1}$, $L_{n-1}, \dots, L_1$ in that order. Indeed, for $1 \le i \le n-1$, δ_i and δ_{2n-1-i} both connect the i^{th} and $(i+1)^{\text{st}}$ punctures of C. Cutting C open along all these arcs, we obtain n components, one of them (corresponding to the first sheet of the covering near $-\infty$) a $(2n-2)$-gon bounded successively by $\delta_1, \delta_2, \dots, \delta_{2n-2}$, while the $n-1$ others (corresponding to sheets $2, \dots, n$ near $-\infty$) are strips bounded by δ_i and δ_{2n-1-i}. From there it is not hard to check that δ_i and δ_{2n-1-i} are both isotopic to L_i for $1 \le i \le n-1$.

The result then follows from Theorem A.1, which asserts that the thimbles $\delta_1, \dots, \delta_{2n-2}$ strictly generate $\mathcal{W}(C)$. ☐

Note that, by this result, L_n could have been omitted entirely from the discussion, as it is quasi-isomorphic to the complex

$$L_1 \xrightarrow{u_{1,2}} L_2 \xrightarrow{u_{2,3}} \cdots \xrightarrow{u_{n-2,n-1}} L_{n-1}.$$

We shall encounter this complex on the mirror side (see Equation (6.2)) in the process of determining the A_∞ structure on the category of matrix factorizations. In particular, we could replace Lemma 4.5 with an argument modeled after that given for Lemma 6.2.

5. The Landau-Ginzburg mirror $(X(n), W)$

In this section we describe mirror Landau-Ginzburg (LG) models $W : X(n) \to \mathbb{C}$ for $n \geq 3$. These mirrors are toric, and their construction can be justified by a physics argument due to Hori and Vafa [20], see also [21, Section 3]. (Mathematically, this construction can be construed as a duality between toric Landau-Ginzburg models.)

Let us start with $\mathbb{P}^1$ minus three points. In this case we can realize our curve as a line in $(\mathbb{C}^*)^2$ viewed as the complement of three lines in $\mathbb{P}^2$. The Hori–Vafa procedure then gives us as mirror LG model a variety $X(3) \subset \mathbb{C}^4$ defined by the equation

$$\mathrm{x}_1\mathrm{x}_2\mathrm{x}_3 = \exp(-t)\mathrm{p}$$

with superpotential $W = \mathrm{p} : X(3) \to \mathbb{C}$, i.e. the mirror LG model $(X(3), W)$ is isomorphic to the affine space $\mathbb{C}^3$ with the superpotential $W = \mathrm{x}_1\mathrm{x}_2\mathrm{x}_3$.

In the case $n = 2k$ we can realize $C = \mathbb{P}^1 \backslash \{2k \text{ points}\}$ as a curve of bidegree $(k-1, 1)$ in the torus $(\mathbb{C}^*)^2$ considered as the open orbit of $\mathbb{P}^1 \times \mathbb{P}^1$. The raw output of the Hori–Vafa procedure is a singular variety $Y(2k) \subset \mathbb{C}^5$ defined by the equations

$$\begin{cases} \mathrm{y}_1 \cdot \mathrm{y}_4 = \mathrm{y}_3^{k-1} \\ \mathrm{y}_2 \cdot \mathrm{y}_5 = \mathrm{y}_3 \end{cases}$$

with y_3 as a superpotential. The variety $Y(2k)$ is a 3-dimensional affine toric variety with coordinate algebra $\mathbb{C}[\mathrm{y}_1, \mathrm{y}_2, \mathrm{y}_3\mathrm{y}_2^{-1}, \mathrm{y}_3^{k-1}\mathrm{y}_1^{-1}]$. A smooth mirror $(X(2k), W)$ can then be obtained by resolving the singularities of $Y(2k)$. More precisely, $Y(2k)$ admits toric small resolutions. Any two such resolutions are related to each other by flops, and thus yield LG models which are equivalent, in the sense that they have equivalent categories of D-branes of type B (see [21]).

If n is odd we realize our curve as a curve in the Hirzebruch surface $\mathbb{F}_1$. All the calculations are similar.

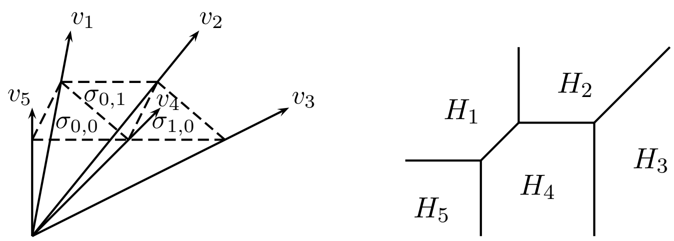

FIGURE 5. The fan Σ and the configuration of divisors H_i (for $n = 5$)

Now we describe a mirror LG model $(X(n), W)$ directly. Consider the lattice $N = \mathbb{Z}^3$ and the fan Σ_n in N with the following maximal cones:

$$\sigma_{i,0} := \langle (i,0,1), (i,1,1), (i+1,0,1) \rangle, \qquad 0 \le i < \left\lfloor \frac{n-1}{2} \right\rfloor,$$

$$\sigma_{i,1} := \langle (i,1,1), (i+1,1,1), (i+1,0,1) \rangle, \quad 0 \le i < \left\lfloor \frac{n-2}{2} \right\rfloor.$$

Let $X(n) := X_{\Sigma_n}$ be the toric variety corresponding to the fan Σ_n.

We label the one-dimensional cones in Σ_n as follows:

$$\mathrm{v}_i := (i-1,1,1), \quad 1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor, \quad \mathrm{v}_i = (n-i,0,1), \quad \left\lfloor \frac{n}{2} \right\rfloor + 1 \le i \le n.$$

For simplicity, we set $\mathrm{v}_{i-n} := \mathrm{v}_i =: \mathrm{v}_{i+n}$. Also, let $H_i := H_{\mathrm{v}_i} \subset X(n)$ be the toric divisor corresponding to the ray v_i (see Figure 5).

The vector $\xi = (0,0,1) \in M = N^\vee$ is non-negative on each cone of Σ_n, and therefore it defines a function

$$W = W_\xi : X(n) \to \mathbb{C},$$

which will be considered as the superpotential. By construction, $W^{-1}(0) = \bigcup_{i=1}^n H_i$.

The LG model $(X(n), W)$ can be considered as a mirror to $C = \mathbb{P}^1 \setminus \{n \text{ points}\}$, by the argument explained above.

Remark 5.1. The construction of the LG model $(X(n), W)$ can also be motivated from the perspective of the Strominger-Yau-Zaslow conjecture. Here again we think of C as a curve in a toric surface, namely we write $C = \overline{C} \cap (\mathbb{C}^*)^2$, where $\overline{C}$ is a rational curve in either $\mathbb{P}^1 \times \mathbb{P}^1$ (for n even) or the Hirzebruch surface $\mathbb{F}_1$ (for n odd). Then, by the main result of [4], $(X(n), W)$ is an SYZ mirror to the blowup of $(\mathbb{C}^*)^2 \times \mathbb{C}$ along $C \times \{0\}$.

6. THE CATEGORY OF D-BRANES OF TYPE B IN LG MODEL $(X(n), W)$

The aim of this section is to describe the category of D-branes of type B in the mirror symmetric LG model $(X(n), W)$, and to show that it is equivalent to the derived category of the wrapped Fukaya category $\mathcal{W}(C)$ calculated in Section 4.

There are two ways to define the category of D-branes of type B in LG models. Assuming that W has a unique critical value at the origin, the first one is to take the triangulated category of singularities $D_{sg}(X_0)$ of the singular fiber $X_0 = W^{-1}(0)$, which is by definition the Verdier quotient of the bounded derived category of coherent sheaves $D^b(\mathrm{coh}(X_0))$ by the full subcategory of perfect complexes $\mathfrak{Perf}(X_0)$.

The other approach involves matrix factorizations. We can define a triangulated category of matrix factorizations $MF(X,W)$ as follows. First define a category $MF^{naive}(X,W)$ whose objects are pairs

$$\underline{T} := \Big(T_1 \underset{t_0}{\overset{t_1}{\rightleftarrows}} T_0 \Big),$$

where T_1, T_0 are locally free sheaves of finite rank on X, and t_1 and t_0 are morphisms such that both compositions $t_1 \cdot t_0$ and $t_0 \cdot t_1$ are multiplication by W. Morphisms in the category $MF^{naive}(X,W)$ are morphisms of pairs modulo null-homotopic morphisms, where a morphism of pairs $f : \underline{T} \to \underline{S}$ is a pair of morphisms $f_1 : T_1 \to S_1$ and $f_0 : T_0 \to S_0$ such that $f_1 \cdot t_0 = s_0 \cdot f_0$ and $s_1 \cdot f_1 = f_0 \cdot t_1$, and a morphism f is null-homotopic if there are two morphisms $h_0 : T_0 \to S_1$ and $h_1 : T_1 \to S_0$ such that $f_1 = s_0 h_1 + h_0 t_1$ and $f_0 = h_1 t_0 + s_1 h_0$.

The category $MF^{naive}(X,W)$ can be endowed with a natural triangulated structure. Now, we consider the full triangulated subcategory of acyclic objects, namely the subcategory $Ac(X,W) \subset MF^{naive}(X,W)$ which consists of all convolutions of exact triples of matrix factorizations. We define a triangulated category of matrix factorizations $MF(X,W)$ on (X,W) as the Verdier quotient of $MF^{naive}(X,W)$ by the subcategory of acyclic objects

$$MF(X,W) := MF^{naive}(X,W)/Ac(X,W).$$

This category will also be called triangulated category of D-branes of type B in the LG model (X,W). It is proved in [32] that there is an equivalence

$$MF(X,W) \xrightarrow{\sim} D_{sg}(X_0), \tag{6.1}$$

where the functor (6.1) is defined by the rule $\underline{T} \mapsto \mathrm{Coker}(t_1)$ and we can regard $\mathrm{Coker}(t_1)$ as a sheaf on X_0 due to it being annihilated by W as a sheaf on X.

In this section we use the first approach and work with the triangulated category of singularities $D_{sg}(X_0)$. This category has a natural DG enhancement, which arises as the DG quotient of the natural DG enhancement of $D^b(\mathrm{coh}(X_0))$ by the DG subcategory of perfect complexes $\mathfrak{Perf}(X_0)$. This implies that the triangulated category of singularities $D_{sg}(X_0)$ has a natural minimal A_∞-structure which is quasi-equivalent to the DG enhancement described above. Thus, in the following discussion we will consider the triangulated category of singularities $D_{sg}(X_0)$ with this natural A_∞-structure.

The singular fiber X_0 of W is the union of the toric divisors in $X(n)$. Consider the structure sheaves $E_i := \mathcal{O}_{H_i}$ as objects of the category $D_{sg}(X_0)$.

Theorem 6.1. *Let $(X(n), W)$ be the LG model described above. Then the triangulated category of singularities $D_{sg}(X_0)$ of the singular fiber $X_0 = W^{-1}(0)$ is strictly generated by n objects $E_1, \ldots, E_n$ and there is a natural isomorphism of algebras*

$$\bigoplus_{i,j} \mathrm{Hom}_{D_{sg}(X_0)}(E_i, E_j) \cong \bigoplus_{i,j} A(X_i, X_j),$$

where A is the category defined in (2.3).

Moreover, the A_∞-structure on $\bigoplus_{i,j} \mathrm{Hom}_{D_{sg}(X_0)}(E_i, E_j)$ is strictly homotopic to $m^{(1,1)}$.

Each object $E_i = \mathcal{O}_{H_i}$, being the cokernel of the morphism $\mathcal{O}_{X(n)}(-H_i) \to \mathcal{O}_{X(n)}$, is a Cohen-Macaulay sheaf on the fiber X_0. Hence by Proposition 1.21 of [31] we have

$$\mathrm{Hom}_{D_{sg}(X_0)}(E_i, E_j[N]) \cong \mathrm{Ext}^N_{X_0}(E_i, E_j)$$

for any $N > \dim X_0 = 2$. Since the shift by [2] is isomorphic to identity, this allows us to determine morphisms between these objects in $D_{sg}(X_0)$ by calculating Ext's between them in the category of coherent sheaves. Hence, if $H_i \cap H_j = \emptyset$, then $\mathrm{Hom}^\bullet_{D_{sg}(X_0)}(E_i, E_j) = 0$.

Assume that $H_i \cap H_j \neq \emptyset$, and denote by Γ_{ij} the curve that is the intersection of H_i and H_j. Consider the 2-periodic locally free resolution of $\mathcal{O}_{H_i}$ on X_0,

$$\{\cdots \longrightarrow \mathcal{O}_{X_0} \longrightarrow \mathcal{O}_{X_0}(-H_i) \longrightarrow \mathcal{O}_{X_0}\} \longrightarrow \mathcal{O}_{H_i} \longrightarrow 0.$$

Now the groups $\mathrm{Ext}^N_{X_0}(E_i, E_j)$ can be calculated as the hypercohomology of the 2-periodic complex

$$0 \longrightarrow \mathcal{O}_{H_j} \xrightarrow{\phi_{ij}} \mathcal{O}_{H_j}(H_i) \xrightarrow{\psi_{ij}} \mathcal{O}_{H_j} \longrightarrow \cdots$$

We first consider the case where $j = i$: then $\phi_{ii} = 0$, and the morphism ψ_{ii} is isomorphic to the canonical map $\mathcal{O}_{H_i}(-D_i) \to \mathcal{O}_{H_i}$, where $D_i = \bigcup_j \Gamma_{ij}$. Hence the cokernel of ψ_{ii} is the structure sheaf $\mathcal{O}_{D_i}$. This implies that $\mathrm{Hom}^\bullet_{D_{sg}(X_0)}(E_i, E_i)$ is concentrated in even degree and the algebra $\mathrm{Hom}^0_{D_{sg}(X_0)}(E_i, E_i)$ is isomorphic to the algebra of regular functions on D_i. However, D_i consists of either two $\mathbb{A}^1$ meeting at one point, two $\mathbb{A}^1$ connected by a $\mathbb{P}^1$, or two $\mathbb{A}^1$ connected by a chain of two $\mathbb{P}^1$ (see Figure 5). In all cases, the algebra of regular functions is isomorphic to $k[x_i, y_i]/(x_i y_i)$.

On the other hand, when $j \neq i$ we must have $\psi_{ij} = 0$, and the cokernel of ϕ_{ij} is isomorphic to $\mathcal{O}_{\Gamma_{ij}}(H_i)$. When $j \notin \{i, i\pm 1\}$ the curve Γ_{ij} is isomorphic to $\mathbb{P}^1$, and moreover the normal bundles to Γ_{ij} in H_i and in H_j are both isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)$. Hence $\mathcal{O}_{\Gamma_{ij}}(H_i) \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ and we obtain that $\mathrm{Hom}^\bullet_{D_{sg}(X_0)}(E_i, E_j)$ is trivial.

When $j = i+1$, the curve Γ_{ij} is isomorphic to $\mathbb{A}^1$ and $\mathrm{Hom}^\bullet_{D_{sg}(X_0)}(E_i, E_j)$ is concentrated in odd degree. Moreover, $\mathrm{Hom}_{D_{sg}(X_0)}(E_i, E_j[1])$ is isomorphic to $H^0(\mathcal{O}_{\Gamma_{ij}})$. Therefore, it is generated by a morphism $u_{i,i+1} : E_i \to E_{i+1}[1]$ as a right module over $\mathrm{End}(E_i)$ and as a left module over $\mathrm{End}(E_{i+1})$, and there are isomorphisms

$$\mathrm{Hom}_{D_{sg}(X_0)}(E_i, E_{i+1}[1]) \cong k[x_{i+1}]u_{i,i+1} = u_{i,i+1}k[y_i].$$

Analogously, if $j = i-1$ then there is a morphism $v_{i,i-1} : E_i \to E_{i-1}[1]$ such that

$$\mathrm{Hom}_{D_{sg}(X_0)}(E_i, E_{i-1}[1]) \cong k[y_{i-1}]v_{i,i-1} = v_{i,i-1}k[x_i].$$

It is easy to check that the composition $v_{i+1,i}u_{i,i+1}$ is equal to y_i and $u_{i,i-1}v_{i,i-1} = x_i$.

Hence, we obtain an isomorphism of super-algebras

$$\bigoplus_{i,j} \mathrm{Hom}_{D_{sg}(X_0)}(E_i, E_j) \cong \bigoplus_{i,j} A(X_i, X_j)$$

This proves the first part of the Theorem.

We claim that the $\mathbb{Z}/2$-graded algebra $\bigoplus_{i,j} \mathrm{Hom}_{D_{sg}(X_0)}(E_i, E_j)$ admits natural lifts to $\mathbb{Z}$-grading, parameterized by vectors $\xi \in N$ such that $\langle \xi, l \rangle = 1$ where $l = (0,0,1)$. Indeed, each such element defines an even grading 2ξ on the algebra $\mathbb{C}[N \otimes \mathbb{C}^*]$ of functions on the torus, with the property that $\deg(W) = 2$. Fixing trivializations of all line bundles restricted to the torus, we then obtain the desired grading. It is easy to check that the resulting grading on cohomology satisfies (2.4).

Now let us calculate the induced A_∞-structure on the algebra $\bigoplus_{i,j} \mathrm{Hom}_{D_{sg}(X_0)}(E_i, E_j)$. By Proposition 3.1 it suffices to compute the numbers

$$a = m_n(u_{i-1,i}, u_{i-2,i-1}, \dots, u_{i,i+1})(0), \quad b = m_n(v_{i+1,i}, v_{i+2,i+1}, \dots, v_{i,i-1})(0).$$

We have $a = b$ by symmetry, and by Remark 3.4 it is sufficient to show that $a \neq 0$.

Lemma 6.2. *In the category $D_{sg}(X_0)$ we have $a = m_n(u_{i-1,i}, u_{i-2,i-1}, \dots, u_{i,i+1})(0) \neq 0$.*

Proof. Consider the complex of objects in the category $D_{sg}(X_0)$:

$$E_1[1-n] \longrightarrow E_2[2-n] \longrightarrow \dots \longrightarrow E_{n-1}[-1], \tag{6.2}$$

where the maps are $u_{i,i+1}$, $1 \le i \le n-2$, and we place $E_{n-1}[-1]$ in degree zero.

The convolution of (6.2) is well defined up to an isomorphism. It is isomorphic to E_n. To see this, introduce the divisor

$$L := \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{k-1}{2} H_k + \sum_{k=\lfloor \frac{n}{2} \rfloor + 1}^{n} \left(\binom{n-k}{2} - 1 \right) H_k.$$

It is straightforward to check that for $i \ge 0$ the restriction of $\mathcal{O}_{X_0}(L - H_1 - \dots - H_i)$ to H_{i+1} is trivial. Moreover, the morphism $u_{i,i+1} : E_i \to E_{i+1}[1]$ for $i \ge 1$ can be interpreted as follows. Let $f : E_i \cong \mathcal{O}_{H_i}(L - H_1 - \cdots - H_{i-1}) \to \mathcal{O}_{\bigcup_{j\neq i} H_j}(L - H_1 - \cdots - H_i)[1]$ be the morphism corresponding to the extension:

$$0 \to \mathcal{O}_{\bigcup_{j\neq i} H_j}(L-H_1-\cdots-H_i) \to \mathcal{O}_{X_0}(L-H_1-\cdots-H_{i-1}) \to \mathcal{O}_{H_i}(L-H_1-\cdots-H_{i-1}) \to 0.$$

Then $Cone(f)$ is a perfect complex, so f is invertible in $D_{sg}(X_0)$. Let g be the projection

$$\mathcal{O}_{\bigcup_{j\neq i} H_j}(L - H_1 - \cdots - H_i)[1] \longrightarrow \mathcal{O}_{H_{i+1}}(L - H_1 - \cdots - H_i)[1].$$

Then $u_{i,i+1} = gf^{-1}$.

By induction, we now see that, for all $1 \leq k \leq n-1$, the following two properties hold:

(1) the convolution $\mathcal{C}_k$ of $E_1[1-n] \xrightarrow{u_{1,2}} E_2[2-n] \longrightarrow \cdots \xrightarrow{u_{k-1,k}} E_k[k-n]$ is isomorphic to $\mathcal{O}_{H_{k+1}\cup\cdots\cup H_n}(L - H_1 - \cdots - H_k)[k+1-n]$, and

(2) the restriction map from $\mathcal{O}_{H_{k+1}\cup\cdots\cup H_n}(L - H_1 - \cdots - H_k)[k+1-n]$ (which is isomorphic to $\mathcal{C}_k$) to $\mathcal{O}_{H_{k+1}}(L - H_1 - \cdots - H_k)[k+1-n] \simeq E_{k+1}[k+1-n]$ corresponds to the morphism $u_{k,k+1} : E_k[k-n] \to E_{k+1}[k+1-n]$.

We conclude that E_n is isomorphic to the convolution $\mathcal{C}_{n-1}$ of (6.2), and that the map from $\mathcal{C}_{n-1}$ to E_n induced by $u_{n-1,n} : E_{n-1}[-1] \to E_n$ is an isomorphism.

Moreover, it is not hard to check that the map from E_n to $\mathcal{C}_{n-1}$ induced by $u_{n,1} : E_n \to E_1$ is also an isomorphism, for instance by using an argument similar to the above one to show that the convolution of

$$E_n[-n] \xrightarrow{u_{n,1}} E_1[1-n] \xrightarrow{u_{1,2}} E_2[2-n] \longrightarrow \cdots \xrightarrow{u_{n-2,n-1}} E_{n-1}[-1]$$

is the zero object.

We claim this implies that $m_n(u_{n-1,n}, u_{n-2,n-1}, \ldots, u_{1,2}, u_{n,1})(0) \neq 0$. The easiest way to see this is to use the language of *twisted complexes* (see e.g. Section 3 of [38]). Recall that twisted complexes are a generalization of complexes in the context of A_∞-categories, for which they provide a natural triangulated enlargement. The philosophy is that, in the A_∞ setting, compositions of maps can only be expected to vanish up to chain homotopies which are explicitly provided as part of the twisted complex; see Section $3l$ of [38] for the actual definition. In our case, the higher compositions of the morphisms within the complex (6.2) are all zero (since the relevant morphism spaces are zero), so (6.2) defines a twisted complex without modification; we again denote this twisted complex by $\mathcal{C}_{n-1}$. Moreover, the maps $u_{n,1}$ and $u_{n-1,n}$ induce morphisms of twisted complexes $\overline{u}_{n,1} \in \mathrm{Hom}^{\mathrm{Tw}}(E_n, \mathcal{C}_{n-1})$ and $\overline{u}_{n-1,n} \in \mathrm{Hom}^{\mathrm{Tw}}(\mathcal{C}_{n-1}, E_n)$, and by the above argument these are isomorphisms. Thus the composition $m_2^{\mathrm{Tw}}(\overline{u}_{n-1,n}, \overline{u}_{n,1})$ is an automorphism of E_n; hence the coefficient of id_{E_n} in this composition is non-zero. However, by definition of the product in the A_∞-category of twisted complexes [38, Equation 3.20],

$$m_2^{\mathrm{Tw}}(\overline{u}_{n-1,n}, \overline{u}_{n,1}) = m_n(u_{n-1,n}, u_{n-2,n-1}, \ldots, u_{1,2}, u_{n,1}).$$

It follows that $a \neq 0$. □

The final ingredient needed for Theorem 6.1 is the following generation statement:

Lemma 6.3. *The objects $E_1, \dots, E_n$ generate the triangulated category $D_{sg}(X_0)$ in the strict sense, i.e. the minimal triangulated subcategory of $D_{sg}(X_0)$ that contains $E_1, \dots, E_n$ coincides with the whole $D_{sg}(X_0)$.*

Proof. Clearly, it suffices to show that the sheaves $\mathcal{O}_{H_1}, \dots, \mathcal{O}_{H_n}$ generate the category $D^b(\mathrm{coh}(X_0))$. Denote by $\mathcal{T} \subset D^b(\mathrm{coh}(X_0))$ the full triangulated subcategory generated by these objects. As above denote by Γ_{st} the intersection $H_s \cap H_t$.

Since the divisors H_s are precisely the irreducible components of X_0, it suffices to prove that that $D^b_{H_s}(\mathrm{coh}(X_0)) \subset \mathcal{T}$ for all $1 \le s \le n$, where $D^b_{H_s}(\mathrm{coh}(X_0))$ is the full subcategory consisting of complexes with cohomology supported on H_s. We introduce a new ordering on the set of components H_s by setting $s_1 = n$, $s_2 = 1$, $s_3 = n-1$, $s_4 = 2$, …, $s_n = \lfloor \frac{n+1}{2} \rfloor$, and will prove by induction on $1 \le i \le n$ that

$$D^b_{H_{s_i}}(\mathrm{coh}(X_0)) \subset \mathcal{T}. \tag{6.3}$$

For $i = 1$ we have $H_{s_1} = H_n \cong \mathbb{A}^2$. Therefore, the sheaf $\mathcal{O}_{H_n}$ generates $D^b(\mathrm{coh}(H_{s_1}))$ and, hence, it generates $D^b_{H_{s_1}}(\mathrm{coh}(X_0))$. Thus, the subcategory $D^b_{H_{s_1}}(\mathrm{coh}(X_0))$ is contained in $\mathcal{T}$. If $n = 3$, then $H_1 \cong H_2 \cong \mathbb{A}^2$, and we are done.

Assume that $n > 3$, and suppose that (6.3) is proved for $1 \le i < k$. By induction hypothesis, $D^b_{\Gamma_{s_j s_k}}(\mathrm{coh}(X_0)) \subset \mathcal{T}$ for any $j < k$. The complement $H_{s_k} \setminus (\bigcup_{j<k} \Gamma_{s_j s_k})$ is isomorphic to either $\mathbb{A}^2$ (if $k < n-1$) or an open subset in $\mathbb{A}^2$ (if $k = n-1$ or n). In any case we obtain that the sheaf $\mathcal{O}_{H_{s_k}}$ together with the subcategories $D^b_{\Gamma_{s_j s_k}}(\mathrm{coh}(X_0))$ for $j < k$ generate $D^b_{H_{s_k}}(\mathrm{coh}(X_0))$. In particular, $D^b_{H_{s_k}}(\mathrm{coh}(X_0)) \subset \mathcal{T}$. This proves (6.3) for $i = k$, which implies that $\mathcal{T} = D^b(\mathrm{coh}(X_0))$. □

7. HMS for cyclic covers

Let d_1, d_2, and d_3 be a triple of integers whose sum is a strictly positive integer D. To this data, we shall associate a trivialization of the tangent space of a D-fold cyclic cover C of $S^2 - \{3 \text{ points}\}$, as well as a Landau-Ginzburg model on an orbifold quotient of $\mathbb{C}^3$. In order to prove that these are mirror, we shall introduce a purely algebraic model for a category equivalent to a full generating subcategory of the Fukaya category on one side and of the category of matrix factorizations on the other, then extend Theorem 1.1 to the cover.

7.1. **A rational grading on A.** The algebraic model corresponds to a choice of a positive integer D, and of integers (p_1, p_2, p_3) and (q_1, q_2, q_3) of parities equal to $D \mod 2$, and whose sum is equal to D. As in Lemma 4.3, we introduce the integers $d_i = \frac{p_i + q_i}{2}$. We also introduce the rational numbers $\tilde{p}_i = p_i/D$, $\tilde{q}_i = q_i/D$, and $\tilde{d}_i = d_i/D$. We then define a $\frac{1}{D}\mathbb{Z}$-*graded* category $A_{(\tilde{p},\tilde{q})}$ (the notation is analogous to that in Definition 2.1) by setting

$$\deg(u_{i-1,i}) = \tilde{p}_i, \tag{7.1}$$

$$\deg(v_{i,i-1}) = \tilde{q}_i. \tag{7.2}$$

Note that additivity with respect to the multiplicative structure determines the rest of the gradings

$$\deg(x_i^k) = \deg(y_{i-1}^k) = 2\tilde{d}_i k, \tag{7.3}$$

$$\deg(x_i^k u_{i-1,i}) = \deg(u_{i-1,i} y_{i-1}^k) = \tilde{p}_i + 2\tilde{d}_i k, \tag{7.4}$$

$$\deg(y_{i-1}^k v_{i,i-1}) = \deg(v_{i,i-1} x_i^k) = \tilde{q}_i + 2\tilde{d}_i k. \tag{7.5}$$

We will now construct from the $\frac{1}{D}\mathbb{Z}$-graded category $A_{(\tilde{p},\tilde{q})}$ a $\mathbb{Z}$-graded category $\tilde{A}_{(\tilde{p},\tilde{q})}$, and discuss A_∞-structures on it. The process we describe is in fact a specific instance of a more general construction (see Definition 7.10).

The first step is to consider an enlargement $\tilde{A}^{[D]}_{(\tilde{p},\tilde{q})}$ of $A_{(\tilde{p},\tilde{q})}$ in which each object is replaced by D different copies, and the groups of morphisms are shifted by multiples of $\frac{1}{D}$. (On the symplectic side, the different objects correspond to the components of the inverse image of a curve under a D-fold covering map.)

$$\mathrm{Ob}\left(\tilde{A}^{[D]}_{(\tilde{p},\tilde{q})}\right) = \{\tilde{X}_i^k \,|\, 0 \leq k < D\} \tag{7.6}$$

$$\tilde{A}^{[D]}_{(\tilde{p},\tilde{q})}(\tilde{X}_i^k, \tilde{X}_j^\ell) = A_{(\tilde{p},\tilde{q})}(X_i, X_j)\left[\frac{2(\ell-k)}{D}\right]. \tag{7.7}$$

Writing $A_{(1,1)}$ for the $\mathbb{Z}/2$-grading on A in which the generators $u_{i-1,i}$ and $v_{i,i-1}$ both have odd degree, we have a forgetful functor

$$\tilde{A}^{[D]}_{(\tilde{p},\tilde{q})} \to A_{(1,1)}$$

which takes $\tilde{X}_i^k$ to X_i. This functor is of course not graded, but there is a maximal subcategory of the source with the property that the restriction becomes a $\mathbb{Z}/2$-graded functor:

Definition 7.1. *The category $\tilde{A}_{(\tilde{p},\tilde{q})}$ has objects those of $\tilde{A}^{[D]}_{(\tilde{p},\tilde{q})}$ and morphisms the subgroup*

$$\tilde{A}_{(\tilde{p},\tilde{q})}(\tilde{X}_i^k, \tilde{X}_j^\ell) \subset A_{(\tilde{p},\tilde{q})}(X_i, X_j)\left[\frac{2(\ell-k)}{D}\right] \tag{7.8}$$

generated by morphisms whose degree is integral, and moreover agrees in parity with the degree of the image in $A_{(1,1)}$.

We shall also need to understand A_∞-structures on $\tilde{A}_{(\tilde{p},\tilde{q})}$. For this, it will be convenient to make the following definition.

Definition 7.2. *A $\frac{1}{D}\mathbb{Z}$-graded A_∞-category B consists of a $\mathbb{Z}/2$-graded A_∞-category B, together with $\frac{1}{D}\mathbb{Z}$-gradings on* $\mathrm{Hom}^{even}(X,Y)$ *and* $\mathrm{Hom}^{odd}(X,Y)$ *for any pair of objects $X, Y \in Ob(B)$, with respect to which the higher products m_n have degree $2-n$.*

A $\frac{1}{D}\mathbb{Z}$-graded DG category is a $\frac{1}{D}\mathbb{Z}$-graded A_∞-category with $m_n = 0$ for $n \geq 3$, and with identity of degree zero; finally, a $\frac{1}{D}\mathbb{Z}$-graded category is a $\frac{1}{D}\mathbb{Z}$-graded DG category with zero differential.

We treat both $A_{(\tilde{p},\tilde{q})}$ and $\tilde{A}^{[D]}_{(\tilde{p},\tilde{q})}$ as $\frac{1}{D}\mathbb{Z}$-graded categories, with $u_{i-1,i}$, $v_{i,i-1}$ being odd morphisms. Note that for a $\frac{1}{D}\mathbb{Z}$-graded A_∞-category B over a field, the standard construction gives a minimal A_∞-structure on the cohomology, i.e. on the $\frac{1}{D}\mathbb{Z}$-graded category $H^*(B)$.

The A_∞-structures of interest to us arise from the fact that any $\frac{1}{D}\mathbb{Z}$-graded A_∞-structure on $A_{(\tilde{p},\tilde{q})}$ extends to $\tilde{A}^{[D]}_{(\tilde{p},\tilde{q})}$, in such a way that $\tilde{A}_{(\tilde{p},\tilde{q})}$ is an A_∞ subcategory. The following result classifies $\frac{1}{D}\mathbb{Z}$-graded A_∞-structures on $A_{(\tilde{p},\tilde{q})}$, by extending Proposition 3.1:

Proposition 7.3. *Equation* (3.1) *gives a bijection between the set of $\frac{1}{D}\mathbb{Z}$-graded A_∞-structures on $A_{(\tilde{p},\tilde{q})}$, up to $\frac{1}{D}\mathbb{Z}$-graded strict homotopy, and k^2.*

Proof. The proof is the same as for Proposition 3.1 (2). Namely, Hochschild cohomology can be defined for $\frac{1}{D}\mathbb{Z}$-graded categories in exactly the same manner as in the $\mathbb{Z}$-graded case, and all the relevant computations from Sections 2 and 3 still hold in this setting. □

Corollary 7.4. *The A_∞ structure on $\tilde{A}_{(\tilde{p},\tilde{q})}$ induced by a $\frac{1}{D}\mathbb{Z}$-graded A_∞-structure on $A_{(\tilde{p},\tilde{q})}$ depends, up to strict $\mathbb{Z}$-graded homotopy, only on the constants a and b appearing in Equation* (3.1)*.*

Proof. A strict homotopy between two A_∞ structures on $A_{(\tilde{p},\tilde{q})}$ extends to one between the structures on $\tilde{A}^{[D]}_{(\tilde{p},\tilde{q})}$. Moreover, if the homotopy is graded, the functor will preserve integral gradings, and hence induce a functor on the integral subcategories. □

The next result will allow us some flexibility in proving homological mirror symmetry by choosing an appropriate graded representative of each object. The key observation needed for its proof is that if we allow arbitrary integers in Equation (7.7), then replacing k by $k+D$ corresponds to a homological shift by 2, so that integrality is preserved as well as parity:

Lemma 7.5. *The closure of $\tilde{A}_{(\tilde{p},\tilde{q})}$ under the shift functor depends, up to isomorphism, only the triple (d_1, d_2, d_3).*

Proof. Let (p'_1, p'_2, p'_3) and (q'_1, q'_2, q'_3) be triples of integers such that

$$p'_i + q'_i = p_i + q_i.$$

The assignment

$$\begin{aligned} X_1^k &\mapsto \tilde{X}_1^k \\ X_2^k &\mapsto \tilde{X}_2^{k+p_2-p'_2} \\ X_3^k &\mapsto \tilde{X}_3^{k+p_2-p'_2+p_3-p'_3} \end{aligned}$$

defines a $\frac{1}{D}\mathbb{Z}$-graded isomorphism, and hence an isomorphism of the corresponding subcategories of integrally graded morphisms. □

7.2. **The wrapped Fukaya category of a cyclic cover.** As in the previous section, we choose integers (d_1, d_2, d_3) whose sum is a strictly positive integer D. Projecting the Riemann surface

$$C = \{(x, y) | y^D = x^{d_2}(1-x)^{d_3}\} \subset \mathbb{C} \times \mathbb{C}^* \tag{7.9}$$

to the x-plane defines a cover of $\mathbb{C} - \{0, 1\}$, in which the punctures are ordered $(\infty, 0, 1)$.

Proposition 7.6. *The wrapped Fukaya category of C, with the $\mathbb{Z}$-grading determined by the restriction of the holomorphic 1-form $\frac{dx}{y}$, is strictly generated by the components of the inverse image of the real axis. Whenever $p_i + q_i = 2d_i$, there is a choice of grading on these components so that the resulting subcategory of the Fukaya category is A_∞-equivalent to the structure induced by $m^{1,1}$ on $\tilde{A}_{(\tilde{p},\tilde{q})}$.*

Remark 7.7. A description of the Fukaya categories of covers as a semi-direct product has previously appeared in the proof of Homological mirror symmetry for the closed genus 2 curve (see [39, Remark 8.1]), and in Sheridan's work [41, Section 7], but our implementation will be quite different because we are concerned with recovering integral gradings that do not come from trivializations of the tangent space of C which are pulled back from the base. Of course, underlying either approach is the fact that each holomorphic disc in the base lifts uniquely, upon choosing a basepoint, to a holomorphic disc in the cover.

In order to prove Proposition 7.6, we choose our curves to be

$$\begin{aligned} L_1 &= (-\infty, 0) \\ L_2 &= (0, 1) \\ L_3 &= (1, +\infty). \end{aligned}$$

Note that each component of the inverse image of L_i in C has constant phase with respect to the 1-form $\frac{dx}{y}$. The different components are distinguished by their phases: those lying over L_2 have phases the D-th roots of unity, while the inverse images of L_1 and L_3 respectively have phases equal to the solutions of $y^D = (-1)^{d_2}$ and $y^D = (-1)^{d_3}$. If we fix the exponential map

$$\alpha \mapsto e^{\pi\sqrt{-1}\alpha}$$

then the graded lifts of such components are again distinguished by the corresponding real-valued phase, which lies in $\frac{d_i}{D} + \frac{2}{D}\mathbb{Z}$. For each integer $0 \leq k < D$, we fix graded lifts $\tilde{L}_i^k$ of L_i with real valued phases

$$\mathrm{Phase}(\tilde{L}_i^k) = \begin{cases} \frac{-d_2}{D} + \frac{2k}{D} & \text{if } i = 1 \\ \frac{2k}{D} & \text{if } i = 2 \\ \frac{d_3}{D} + \frac{2k}{D} & \text{if } i = 3. \end{cases}$$

If we use a Hamiltonian on C which is pulled back from $\mathbb{C} - \{0, 1\}$, a chord between $\tilde{L}_i^k$ and $\tilde{L}_j^\ell$ is uniquely determined by its projection to $\mathbb{C}$, which is a chord with endpoints on L_i and L_j. Choosing the Hamiltonian as in Section 4, the differential in the Floer complex vanishes, so that $HW^*(\tilde{L}_i^k, \tilde{L}_j^\ell)$ is the subgroup of $HW^*(L_i, L_j)$ generated by those chords admitting a lift with the correct boundary conditions.

By construction, we have arranged for the chords $v_{2,1}$ and $u_{2,3}$ to respectively lift to generators of $HW^*(\tilde{L}_2^0, \tilde{L}_1^0)$ and $HW^*(\tilde{L}_2^0, \tilde{L}_3^0)$. It is then not hard to see that the generators of $HW^*(\tilde{L}_2^0, \tilde{L}_1^0)$ correspond to lifts of chords $v_{2,1}x_2^k$ whenever D divides kd_2, while the generators of $HW^*(\tilde{L}_2, \tilde{L}_3)$ are lifts of $y_2^k u_{2,3}$ where D divides kd_3.

Note that if we set $q_2 = p_3 = D$, these are precisely the monomials in $A_{(\tilde{p},\tilde{q})}(X_2, X_1)$ and $A_{(\tilde{p},\tilde{q})}(X_2, X_3)$ of odd integer degree, i.e. the generators of $\tilde{A}_{(\tilde{p},\tilde{q})}(\tilde{X}_2^0, \tilde{X}_1^0)$ and $\tilde{A}_{(\tilde{p},\tilde{q})}(\tilde{X}_2^0, \tilde{X}_3^0)$. Extending this computation from $k = \ell = 0$ to the general case, and using the fact that a holomorphic curve in $\mathbb{C} - \{0, 1\}$ lifts uniquely to C upon choosing a basepoint, we conclude:

Lemma 7.8. *If $(p_1, p_2, p_3) = (D - 2d_2, 2d_2 - D, D)$ and $(q_1, q_2, q_3) = (D - 2d_3, D, 2d_3 - D)$, then the subcategory of $\mathcal{W}(C)$ with objects $\tilde{L}_i^k$ is quasi-isomorphic to $\tilde{A}_{(\tilde{p},\tilde{q})}$ equipped with the A_∞ structure induced by $m^{1,1}$.* □

This result, together with Lemma 7.5, implies the second part of Proposition 7.6, while the first part follows from Theorem A.1 applied to the composition of the covering map from C to $\mathbb{C} - \{0, 1\}$ with the Lefschetz fibration used in Lemma 4.6.

7.3. **Equivariant Landau-Ginzburg mirror model.** Consider $\mathbb{C}^3$ equipped with the diagonal action of $G = \mathbb{Z}/D$ with weights $\frac{1}{D}(d_1, d_2, d_3)$, where $d_i = \frac{p_i + q_i}{2}$ as above. Let $W := z_1 z_2 z_3 \in \mathbb{C}[z_1, z_2, z_3]^G$. Our LG model is $(\mathbb{C}^3//G, W)$. We have an equivalence

$$D_{sg}^G(W^{-1}(0)) \cong MF^G(W). \tag{7.10}$$

For each $\chi \in G^* \cong \mathbb{Z}/D$, we have a functor $-(\chi)$ on $D_{sg}(W^{-1}(0))$. For each $0 \le k < D$, denote by $\chi_k \in G^*$ the character corresponding to the image of k in $\mathbb{Z}/D$. Take the objects

$$E_i^k := \mathcal{O}_{H_i}(\chi_k) \in D_{sg}^G(W^{-1}(0)), \quad 1 \le i \le 3, \quad 0 \le k < D,$$

where $H_i = \{z_i = 0\} \subset W^{-1}(0)$. Clearly, they generate (strictly) the category $D_{sg}^G(W^{-1}(0))$. Now we would like to prove that there is an equivalence $D\mathcal{W}(C) \cong D_{sg}^G(W^{-1}(0))$, such that the objects $\tilde{L}_i^k$ correspond to E_i^k. To do that, we will deal with $\frac{1}{D}\mathbb{Z}$-gradings on matrix factorizations.

Put $\deg(z_i) := 2\tilde{d}_i = \frac{2d_i}{D}$. Then the algebra $R = \mathbb{C}[z_1, z_2, z_3]$ becomes $\frac{1}{D}\mathbb{Z}$-graded, and $\deg(W) = 2$. Define a $\frac{1}{D}\mathbb{Z}$-graded DG category $MF^{\frac{1}{D}\mathbb{Z}}(W)$ of $\frac{1}{D}\mathbb{Z}$-graded matrix factorizations as follows.

An object of this category is a pair of free finitely generated $\frac{1}{D}\mathbb{Z}$-graded R-modules $\underline{T} = (T_1, T_0)$, together with homogeneous morphisms $t_1 : T_1 \to T_0$, $t_0 : T_0 \to T_1$ of degree 1, such that $t_1 t_0 = W \cdot \mathrm{id}_{T_0}$, $t_0 t_1 = W \cdot \mathrm{id}_{T_1}$.

Further, for two objects $\underline{T}, \underline{S}$, the 2-periodic complex of morphisms $\mathrm{Hom}(\underline{T}, \underline{S})$ is defined as usual. Composition is also the usual one. Finally, the $\frac{1}{D}\mathbb{Z}$-grading on $\mathrm{Hom}^{even}(\underline{T}, \underline{S})$ and $\mathrm{Hom}^{odd}(\underline{T}, \underline{S})$ comes from the $\frac{1}{D}\mathbb{Z}$-gradings on T_1, T_0, S_1, S_0.

It is straightforward to check that we get indeed a $\frac{1}{D}\mathbb{Z}$-graded DG category. Now we consider three particular matrix factorizations $\underline{T}_1, \underline{T}_2, \underline{T}_3 \in MF^{\frac{1}{D}\mathbb{Z}}(W)$ as follows:

$$\underline{T}_1 = \{R \xrightarrow{z_2 z_3} R[1 - 2\tilde{d}_1] \xrightarrow{z_1} R\},$$

and analogously for $\underline{T}_2, \underline{T}_3$. Denote by $\mathcal{C}_{d_1,d_2,d_3} \subset MF^{\frac{1}{D}\mathbb{Z}}(W)$ the full $\frac{1}{D}\mathbb{Z}$-graded DG subcategory with objects $\underline{T}_1, \underline{T}_2, \underline{T}_3$. Then the $\frac{1}{D}\mathbb{Z}$-graded cohomological category $H^*(\mathcal{C}_{d_1,d_2,d_3})$ is equipped with a natural minimal A_∞-structure (defined up to graded strict homotopy).

For convenience, set $\underline{T}_{i+3} := \underline{T}_i$, $z_{i+3} := z_i$, and $d_{i+3} := d_i$.

Proposition 7.9. (1) *There is a natural equivalence of $\frac{1}{D}\mathbb{Z}$-graded categories $A_{(\tilde{p},\tilde{q})} \cong H^*(\mathcal{C}_{d_1,d_2,d_3})$, where $p_i = 2d_i + 2d_{i+1} - D$ and $q_i = 2d_{i-2} + 2d_{i-1} - D$.*

(2) *Under the above equivalence, the A_∞-structure on $H^*(\mathcal{C}_{d_1,d_2,d_3})$ is homotopic to $m^{1,1}$.*

Proof. (1) For each $i = 1, 2, 3$, consider the odd closed morphism $\tilde{u}_{i-1,i} : \underline{T}_{i-1} \to \underline{T}_i$ given by the pair of morphisms

$$R \xrightarrow{z_{i+1}} R[1 - 2\tilde{d}_i], \quad R[1 - 2\tilde{d}_{i-1}] \xrightarrow{-1} R.$$

The sign appears because the morphism is odd. Clearly, $\deg(\tilde{u}_{i-1,i}) = \frac{p_i}{D} = \tilde{p}_i$. Similarly, consider the odd morphism $\tilde{v}_{i,i-1} : \underline{T}_i \to \underline{T}_{i-1}$ given by the pair of morphisms:

$$R \xrightarrow{z_{i-2}} R[1 - 2\tilde{d}_{i-1}], \quad R[1 - 2\tilde{d}_i] \xrightarrow{-1} R.$$

It is easy to see that $\deg(\tilde{v}_{i,i-1}) = \tilde{q}_i$. Moreover, the compositions $\tilde{u}_{i+1,i}\tilde{u}_{i-1,i}$ and $\tilde{v}_{i,i-1}\tilde{v}_{i+1,i}$ are homotopic to zero. Hence, we have a functor

$$A_{(\tilde{p},\tilde{q})} \longrightarrow H^*(\mathcal{C}_{d_1,d_2,d_3})$$

of $\frac{1}{D}\mathbb{Z}$-graded categories. It is easily checked to be an equivalence.

(2) The non-vanishing of the constant terms of $m_3(\tilde{u}_{3,1}, \tilde{u}_{2,3}, \tilde{u}_{1,2})$ and $m_3(\tilde{v}_{2,1}, \tilde{v}_{3,2}, \tilde{v}_{1,3})$ follows from the results of Section 6. Indeed these constants terms do not depend on gradings, and they were shown not to vanish for integer gradings. Hence, the statement follows from Proposition 7.3. ☐

Definition 7.10. *For a $\frac{1}{D}\mathbb{Z}$-graded A_∞-category B, denote by $\tilde{B}$ the $\mathbb{Z}$-graded A_∞-category whose objects are pairs (X, k), where $X \in Ob(B)$ and $0 \le k < D$, and where morphisms are defined by the formula*

$$\mathrm{Hom}^{2i}_{\tilde{B}}((X,k),(Y,l)) = \mathrm{Hom}^{2i+\frac{2(l-k)}{D},even}(X,Y)$$

$$\mathrm{Hom}^{2i-1}_{\tilde{B}}((X,k),(Y,l)) = \mathrm{Hom}^{2i-1+\frac{2(l-k)}{D},odd}(X,Y).$$

The higher products are induced by those of B.

(Compare with the construction in Section 7.1.)

It is clear that the assignment $B \mapsto \tilde{B}$ defines a functor from $\frac{1}{D}\mathbb{Z}$-graded A_∞-categories and A_∞-morphisms to usual $\mathbb{Z}$-graded A_∞-categories and A_∞-morphisms.

Corollary 7.11. *With the same notation, the DG category $\widetilde{\mathcal{C}_{d_1,d_2,d_3}}$ is quasi-equivalent to the A_∞-category $(\tilde{A}_{(\tilde{p},\tilde{q})}, \tilde{m}^{1,1})$, where the A_∞-structure $\tilde{m}^{1,1}$ is induced by $m^{1,1}$.*

Now write the matrix factorizations in $MF^G(W)$ corresponding to the above generators $E_i^k \in D^G_{sg}(W^{-1}(0))$:

$$\tilde{\underline{T}}{}_i^k = \{R(\chi_k) \xrightarrow{z_{i+1}z_{i+2}} R(\chi_{k-d_i}) \xrightarrow{z_i} R(\chi_k)\}.$$

Then it is straightforward to see that we have a fully faithful functor of $\mathbb{Z}/2$-graded DG categories

$$\widetilde{\mathcal{C}_{d_1,d_2,d_3}} \longrightarrow MF^G(W), \quad (\underline{T}_i, k) \mapsto \tilde{\underline{T}}{}_i^k.$$

Since the collection of sheaves $\{\mathcal{O}_{H_i}(\chi_k)\}_{k=0}^{D-1}$ strongly generate the category of equivariant coherent sheaves on $W^{-1}(0)$ supported on the component H_i, we obtain the following result using the same argument as the proof of Lemma 6.3:

Proposition 7.12. *The triangulated category $D^G_{sg}(W^{-1}(0))$ is strictly generated by the objects E_i^k introduced above. The resulting $\mathbb{Z}/2$-graded DG subcategory of $MF^G(W)$ is quasi-equivalent to the ($\mathbb{Z}/2$-graded) A_∞-category $\tilde{A}_{(\tilde{p},\tilde{q})}$.*

Taking into account the results of the previous Subsection, we have proved the following theorem.

Theorem 7.13. *The triangulated categories $D\mathcal{W}(C)$ and $D^G_{sg}(W^{-1}(0))$ are equivalent.*

Proof. This follows from Proposition 7.12, Proposition 7.6 and Lemma 7.5. □

Appendix A. A generation result for the wrapped Fukaya category

Throughout this section, we shall consider $\pi\colon \Sigma \to D^2$, a Lefschetz fibration on a compact Riemann surface with boundary, i.e. a simple branched covering of the disc. The inverse image of an arc starting at a critical value and ending at $1 \in D^2$ is called a Lefschetz thimble, and the collection of thimbles obtained by choosing a collection of arcs which do not intersect in the interior, one for each critical point, is called a *basis of thimbles.*

Theorem A.1. *Any basis of thimbles generates (in the strict sense) the wrapped Fukaya category of Σ for all coefficient rings.*

Note that this result is stronger than the split-generation statement that might be expected by applying the results of [2]. We shall prove it by embedding Σ inside a larger Riemann surface where the Lagrangians we consider extend to circles. Then, following the strategy developed by Seidel in [38], we apply the long exact sequence for a Dehn twist to derive a generation statement in the Fukaya category of *compact* Lagrangians. Finally, we shall use the existence of a restriction functor constructed in [6] to conclude the desired result. We shall omit discussions of signs and gradings (and the corresponding geometric choices) which essentially play no role in our arguments.

Let us therefore start by choosing a Liouville structure on Σ, i.e. a 1-form λ whose differential is symplectic, and whose associated Liouville flow is outward pointing at the boundary.

In addition to mere exactness, the construction of a restriction functor will require us to consider the following technical condition on a curve $\alpha \in \Sigma$

(A.1) $\lambda|\alpha$ has a primitive function which vanishes on the boundary.

Choosing a basis of thimbles, we replace λ (adding the differential of a function) so that this condition holds for each element of the basis. For more general curves, we have:

Lemma A.2. *Every exact curve in Σ is equivalent, in the wrapped Fukaya category, to a curve satisfying Condition* (A.1)*.*

Proof. The quasi-isomorphism class of a curve is invariant under Hamiltonian isotopies in the completion of Σ to a surface of infinite area. We leave the (easier) non-separating case to the reader, and assume we are given a curve α_0 whose union with a subset of $\partial\Sigma$ (consisting of an interval together with some components) bounds a submanifold Σ_0. Stokes's theorem implies that difference between the values of a primitive at the two endpoints of α_0 equals

$$\int_{\Sigma_0} \omega - \int_{\Sigma_0 \cap \partial\Sigma} \lambda$$

where each component of $\Sigma_0 \cap \partial\Sigma$ is given the orientation induced as a subset of the boundary of Σ. Note that the integral over the boundary is strictly greater than 0 and smaller than the area of Σ. In particular, we may isotope α_0, through embedded curves which have the same boundary, to a curve α_1 bounding a surface Σ_1 of area exactly $\int_{\Sigma_0 \cap \partial\Sigma} \lambda$. Stokes's theorem now implies that any primitive on α_1 must have equal values at the endpoints. The isotopy between α_0 and α_1 can be made Hamiltonian after enlarging Σ by attaching infinite cylinders to its boundary components. □

To prove that thimbles generate the wrapped Fukaya category, it suffices therefore to prove that an arbitrary curve γ, satisfying Condition (A.1), is equivalent to an iterated cone built from thimbles. We consider the Riemann surface Σ_γ obtained by attaching a 1-handle along the boundary of γ. Weinstein's theory of handle attachment gives a Liouville form on Σ_γ for which the inclusion of Σ is a subdomain, and such that the union of γ with the core of the new handle is an exact Lagrangian circle which we shall denote γ_0. In addition, we may construct a Lefschetz fibration

$$\pi_\gamma \colon \Sigma_\gamma \to D^2(1+\epsilon)$$

over the disc of radius $1+\epsilon$, whose restriction to Σ agrees with π, and which has exactly one critical point outside the unit disc.

Let us choose a basis of thimbles for π_γ extending the previous basis, and such that the additional arc does not enter the unit disc. We then consider a double cover of Σ_γ denoted $\tilde{\Sigma}_\gamma$, which is branched at the inverse image of $1+\epsilon$. The thimbles of π_γ double to exact Lagrangian circles $(\gamma_1, \dots, \gamma_d, \gamma_{d+1})$ in $\tilde{\Sigma}_\gamma$, with the convention that γ_{d+1} is the double of the thimble coming from the new critical point. Since γ_0 does not link the branching point, its inverse image in $\tilde{\Sigma}_\gamma$ consists of a pair of curves which we shall denote $\gamma_\pm$.

The following result is essentially Proposition 18.15 of [38]. Its proof relies on the correspondence between algebraic and geometric Dehn twists, and the fact that applying a series of Dehn twists about the curves $\gamma_1, \dots, \gamma_{d+1}$ maps γ_+ to a curve isotopic to γ_-.

Lemma A.3. *The direct sum of γ_+ with an object geometrically supported on γ_- lies in the category generated by $(\gamma_1, \dots, \gamma_d, \gamma_{d+1})$.* □

Proposition 18.15 of [38] in fact describes the precise object supported on γ_- which appears in this Lemma; as this is inconsequential for our intended use we avail ourselves of the option of omitting any discussion of signs and gradings. We complete this appendix with the proof of its main result:

Proof of Theorem A.1. The inverse image of Σ in $\tilde{\Sigma}_\gamma$ consists of two components; by fixing the one including γ_+, we obtain an inclusion

$$\iota\colon \Sigma \to \tilde{\Sigma}_\gamma,$$

which is again an inclusion of Liouville subdomains for an appropriate choice of Liouville form on the double branched cover.

By construction, γ_- and γ_{d+1} are disjoint from $\iota(\Sigma)$, while γ_+ intersects $\iota(\Sigma)$ in γ and $(\gamma_1, \dots, \gamma_d)$ in the originally chosen basis of vanishing cycles. Since, by construction, Condition (A.1) holds for these curves, we may apply the restriction functor defined in Sections 5.1 and 5.2 of [6]. This A_∞ functor, defined on the subcategory of the Fukaya category of $\tilde{\Sigma}_\gamma$ consisting of objects supported on one of the curves $(\gamma_+, \gamma_-, \gamma_1, \dots, \gamma_d)$, has target the wrapped Fukaya category of $\iota(\Sigma)$ and takes a curve to its intersection with the subdomain. By Lemma A.3, the direct sum of γ_+ and an object supported on γ_- lies in the category by generated by $(\gamma_1, \dots, \gamma_d, \gamma_{d+1})$. Since γ_- is disjoint from $\iota(\Sigma)$, the image of this direct sum under restriction is γ, so we conclude, as desired, that γ lies in the category generated by thimbles. □

Appendix B. Idempotent completion

The purpose of this appendix is to prove that the triangulated category of singularities $D_{sg}(X_0)$ of the singular fiber $X_0 = W^{-1}(0)$ of the LG model $(X(n), W)$ is idempotent complete. This implies that the derived wrapped Fukaya category $D\mathcal{W}(C)$ is also idempotent complete.

A full triangulated subcategory $\mathcal{N}$ of a triangulated category $\mathcal{T}$ is called *dense* in $\mathcal{T}$ if each object of $\mathcal{T}$ is a direct summand of an object isomorphic to an object in $\mathcal{N}$. An amazing theorem of R. Thomason [43, Th. 2.1] asserts that there is an one-to-one correspondence between strictly full dense triangulated subcategories $\mathcal{N}$ in $\mathcal{T}$ and subgroups H of the Grothendieck group $K_0(\mathcal{T})$. Moreover, we know that under this correspondence $\mathcal{N}$ goes to the image of $K_0(\mathcal{N})$ in $K_0(\mathcal{T})$ and to H we attach the full subcategory $\mathcal{N}_H$ whose objects are those N in $\mathcal{T}$ such that $[N] \in H \subset K_0(\mathcal{T})$. Actually, in this situation map from $K_0(\mathcal{N})$ to $K_0(\mathcal{T})$ is an inclusion.

Let us consider the triangulated category of singularities $D_{sg}(Z)$ for some scheme Z. The Grothendieck group $K_0(D_{sg}(Z))$ is equal to the cokernel of the map $K_0(\mathfrak{Perf}(Z)) \to K_0(D^b(\operatorname{coh} Z))$.

On the other hand, by [35, Th. 9] (see also [24, Th. 5.1]) there is a long exact sequence for K-groups

$$\cdots \longrightarrow K_i(\mathfrak{Perf}(Z)) \longrightarrow K_i(D^b(\operatorname{coh} Z)) \longrightarrow K_i(\overline{D_{sg}(Z)}) \longrightarrow K_{i-1}(\mathfrak{Perf}(Z)) \longrightarrow \cdots$$

where $\overline{D_{sg}(Z)}$ is the idempotent closure (or *Karoubian completion*) of $D_{sg}(Z)$.

Using the fact that K_{-1} is trivial for a small abelian category ([35, Th. 6]), we obtain a short exact sequence

$$0 \longrightarrow K_0(D_{sg}(Z)) \longrightarrow K_0(\overline{D_{sg}(Z)}) \longrightarrow K_{-1}(\mathfrak{Perf}(Z)) \longrightarrow 0.$$

This sequence shows that $K_{-1}(\mathfrak{Perf}(Z))$ is a measure of the difference between $D_{sg}(Z)$ and its idempotent completion $\overline{D_{sg}(Z)}$.

To summarize all these results, we obtain the following proposition:

Proposition B.1. *The triangulated category of singularities $D_{sg}(Z)$ is idempotent complete if and only if $K_{-1}(\mathfrak{Perf}(Z)) = 0$.*

Also recall that the negative K-groups are defined by induction from the following exact sequences

$$0 \to K_i(\mathfrak{Perf}(Z)) \to K_i(\mathfrak{Perf}(Z[t])) \oplus K_i(\mathfrak{Perf}(Z[t^{-1}])) \to K_i(\mathfrak{Perf}(Z[t, t^{-1}])) \to \\ \to K_{i-1}(\mathfrak{Perf}(Z)) \to 0.$$

In particular, the group $K_{-1}(\mathfrak{Perf}(Z))$ is isomorphic to the cokernel of the canonical map $K_0(\mathfrak{Perf}(Z[t])) \oplus K_0(\mathfrak{Perf}(Z[t^{-1}])) \to K_0(\mathfrak{Perf}(Z[t, t^{-1}]))$.

Now we consider the specific case $Z = X_0$, where X_0 is the singular fiber of $W : X(n) \to \mathbb{C}$ defined in Section 5, i.e. the union of the toric divisors of $X(n)$.

Proposition B.2. *Let X_0 be as above, then $K_{-1}(\mathfrak{Perf}(X_0)) = 0$.*

Proof. Let us denote by $\Gamma \subset X_0$ the one-dimensional subscheme consisting of the singularities of X_0, i.e. the union of all the toric curves in $X(n)$. Denote by $\pi : X_0' \to X_0$ the normalization of X_0 and set $\Gamma' = \Gamma \times_{X_0} X_0'$. By [45, Th. 3.1] there is a long exact sequence of K-groups which in this case gives the following exact sequence:

$$K_0(X_0') \oplus K_0(\Gamma) \longrightarrow K_0(\Gamma') \longrightarrow K_{-1}(X_0) \longrightarrow K_{-1}(X_0') \oplus K_{-1}(\Gamma),$$

where all K-groups are K-groups of perfect complexes. Since the normalization X_0' is the disjoint union of smooth toric surfaces we have $K_{-1}(X_0') = 0$. Considering components of the normalization X_0' it is also easy to deduce that the restriction map $K_0(X_0') \to K_0(\Gamma')$ is surjective. Thus it is sufficient to show that $K_{-1}(\Gamma)$ is trivial.

To any Noetherian curve C we can associate a bipartite graph γ defined as follows. The graph γ has one vertex for each singular point s of C and one vertex for each component of the normalization $p: C' \to C$. For each point of $p^{-1}(s)$ there is an edge connecting the corresponding component of C' with the singular point s of C.

By [45, Lemma 2.3] there is an isomorphism $K_{-1}(C) = \mathbb{Z}^{\lambda}$, where λ is the number of loops in the bipartite graph γ associated to C. It is easy to see that in our case the bipartite graph of Γ does not have any loop. Thus $K_{-1}(\Gamma) = 0$, and $K_{-1}(X_0) = 0$ too. □

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