Wall-crossing structures

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WCF and MS

In 2004 in a joint work with Maxim Kontsevich we introduced wall-crossing formulas (WCF) for constructing Calabi-Yau mirror duals in SYZ approach to Mirror Symmetry. The base B of SYZ torus fibration is divided into pieces by codim 1 walls which parametrize torus fibers containing boundaries of pseudo-holomorphic discs. Generic points of the walls are endowed with elements of certain pronilpotent groups. Those group elements depend on the moduli spaces of discs. Then WCF is a compatibility condition which says that the product of the group elements along a generic small closed loop is *id*.

WCF and DT-invariants

Similar WCFs occur in our 2008 paper on Donaldson-Thomas invariants. In that case $B \subset Stab(\mathcal{C})$, where $\mathcal{C}$ is a $3CY$ category. Point $b \in B$ belongs to a wall if the corresponding central charge Z_b maps a sublatice of rank 2 in $K_0(\mathcal{C})$ into a straight line. Group elements assigned to generic points of the walls are defined in terms of DT-invariants of $\mathcal{C}$ associated with the stability condition b. They are virtual numbers of semistable objects.

In 2008 paper arXiv: 0811.2435 we proposed that DT-invariants encode a geometric object which is a (formal) Poisson manifold. And moreover they can be reconstructed from this Poisson manifold.

Thus we have two stories which are similar: in both cases (DT and MS) we would like to construct a Poisson (maybe symplectic) manifold using WCFs. And the "input" geometry is similar: affine (or even linear) manifold divided into pieces by real codimension 1 walls, which are endowed with group elements or more generally, group-valued piecewise-linear functions.

There are many other situations (besides of DT-theory of $3CY$ categories and MS) where WCF appear. E.g.:

1) Physics:

a) $N = 2, d = 4$ gauge theories;

b) Supersymmetric black holes in supergravity.

2) Mathematics:

a) Complex integrable systems of Hitchin type;

b) Resurgence of WKB approximations of differential equations with small parameter;

c) Counting of geodesics of quadratic differentials on curves;

d) Deformation theory of a wheel of projective lines in a Poisson manifold (related to clusters);

and many others...

In a joint work with M.K. (will appear soon) we suggested that in all examples there is an underlying mathematical structure, which we call **Wall-Crossing Structure** (WCS).

Aim of the talk: discuss the generalities of WCS as well as mathematical questions arising in the course of its construction in DT-theory, complex integrable systems, Calabi-Yau 3-folds, deformation theory of the wheel of projective lines.

Mirror Symmetry case is slightly different. If time permits, I am going to discuss the corresponding analog of WCS.

Set up in the nilpotent case

- Γ denotes a fixed finitely-generated free abelian group, i.e. $\Gamma \simeq \mathbf{Z}^k$ for some $k \in \mathbf{Z}_{\geq 0}$. The associated real vector space is $\Gamma_{\mathbf{R}} := \Gamma \otimes \mathbf{R}$. We will denote by $\mathfrak{g}$ a fixed Γ-graded Lie algebra over $\mathbf{Q}$,

$$\mathfrak{g} = \bigoplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma.$$

Let us assume that the set

$$\text{Supp } \mathfrak{g} := \{\gamma \in \Gamma \,|\, \mathfrak{g}_\gamma \neq 0\} \subset \Gamma$$

is finite and is contained in an open half-space in $\Gamma_{\mathbf{R}}$. In particular, all elements of Supp $\mathfrak{g}$ are non-zero, i.e. $\mathfrak{g}_0 = 0$. Under our assumption the Lie algebra $\mathfrak{g}$ is nilpotent. Let us denote by G the corresponding nilpotent group. The exponential map $\exp : \mathfrak{g} \to G$ is a bijection of sets.

Walls

The finite union of hyperplanes $\gamma^\perp \subset \Gamma_{\mathbf{R}}^*$ ("wall associated with γ") we will denote by $\mathrm{Wall}_{\mathfrak{g}}$. Its complement has a finite number of connected components which are open convex domains in $\Gamma_{\mathbf{R}}^*$. These components are exactly open strata in the natural stratification of $\Gamma_{\mathbf{R}}^*$ associated with the finite collection of hyperplanes $\left(\gamma^\perp\right)_{\gamma \in \mathrm{Supp}\,\mathfrak{g}}$. Notice that different elements $\gamma \in \mathrm{Supp}(\mathfrak{g})$ can give the same hyperplane,

$$\gamma_1^\perp = \gamma_2^\perp \iff \gamma_1 \parallel \gamma_2.$$

Definition

A (global) wall-crossing structure (I abbreviate it as (global) WCS for short) for $\mathfrak{g}$ is an assignment

$$(y_1, y_2) \to g_{y_1, y_2} \in G$$

for any $y_1, y_2 \in \Gamma_{\mathbf{R}}^* - \mathrm{Wall}_{\mathfrak{g}}$ which is locally constant in y_1, y_2, satisfies the cocycle condition

$$g_{y_1, y_2} \cdot g_{y_2, y_3} = g_{y_1, y_3} \quad \forall y_2, y_2, y_3 \in \Gamma_{\mathbf{R}}^* - \mathrm{Wall}_{\mathfrak{g}}$$

and such that in the case when the straight interval connecting y_1 and y_2 intersects only one of hyperplanes $\gamma^{\perp}$ then

$$\log(g_{y_1, y_2}) \in \bigoplus_{\gamma' : \gamma' \| \gamma} \mathfrak{g}_{\gamma'} .$$

WCS as a group element

Notice that the complement $\Gamma_{\mathbf{R}}^* - \mathrm{Wall}_{\mathfrak{g}}$ contains two distinguished components U_+, U_- (which are different iff $\mathfrak{g} \neq 0$) consisting of points $y \in \Gamma_{\mathbf{R}}^*$ such that $y(\gamma) > 0$ (resp. $y(\gamma) < 0$) for all $\gamma \in \mathrm{Supp}\,\mathfrak{g}$. Hence with any global WCS $\sigma = (g_{y_1,y_2})$ we can associate an element

$$g_{+,-} := g_{y_+,y_-} \in G, \ \ y_\pm \in U_\pm.$$

One can prove that the map $\sigma \mapsto g_{+,-}$ provides a bijection between the set of wall-crossing structures and G (considered as a set).
For any point $y \in \Gamma_{\mathbf{R}}^*$ we have a decomposition of $\mathfrak{g}$ (considered as a vector space) into the direct sum of three vector spaces

$$\mathfrak{g} = \mathfrak{g}_-^{(y)} \oplus \mathfrak{g}_0^{(y)} \oplus \mathfrak{g}_+^{(y)}$$

corresponding to components $\mathfrak{g}_\gamma$ such that $y(\gamma) \in \mathbf{R}$ is negative, zero or positive respectively.

Factorization into a product of three and the sheaf

We denote by $G_-^{(y)}, G_0^{(y)}, G_+^{(y)}$ the corresponding nilpotent subgroups of G. Then it is easy to see that the multiplication map

$$G_-^{(y)} \times G_0^{(y)} \times G_+^{(y)} \to G \quad , (g_-, g_0, g_+) \mapsto g_- \cdot g_0 \cdot g_+$$

is a bijection. Hence any element $g \in G$ can be uniquely decomposed as the product

$$g = g_-^{(y)} g_0^{(y)} g_+^{(y)}.$$

We denote by $\pi_y : G \to G_0^{(y)} = G_-^{(y)} \backslash G / G_+^{(y)}$ the canonical projection to the double coset. In the above notation we have $\pi_y(\mathfrak{g}) = \mathfrak{g}_0^{(y)}$.

Claim: There exists a sheaf of sets on $\Gamma_{\mathbf{R}}^*$ with the stalk over $y \in \Gamma_{\mathbf{R}}^*$ given by $G_0^{(y)}$. It is called **the sheaf of wall-crossing structures** and is denoted by $WCS_{\mathfrak{g}}$.

Cones in the pronilpotent case

Since typically there are infinitely many walls, we need to generalize the above story.

Let $\mathfrak{g} = \oplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ be a graded Lie algebra. We do not impose any restrictions on $\mathrm{Supp}(\mathfrak{g})$.

In order to define WCS in this case we fix a convex cone $C \subset \Gamma_{\mathbf{R}} := \Gamma \otimes \mathbf{R}$ such that the closure of C does not contain a line (*strict cone*). Yet another equivalent condition: there exists $\phi \in \Gamma_{\mathbf{R}}^*$ such that the restriction of ϕ to the cone C is a proper map to $\mathbf{R}_{\geq 0}$.

In this case we define a pronilpotent Lie algebra $\mathfrak{g}_C$ as an infinite product

$$\mathfrak{g}_C := \prod_{\gamma \in C \cap \Gamma - \{0\}} \mathfrak{g}_\gamma$$

and denote by G_C the corresponding pronilpotent group. The exponential map identifies $\mathfrak{g}_C$ and G_C.

Lie algebra $\mathfrak{g}_C$ is the projective limit of nilpotent Lie algebras

$$\mathfrak{g}_{C,\phi}^{(k)} = \oplus_{\gamma \in C \cap \Gamma - \{0\}, \phi(\gamma) \le k} \mathfrak{g}_\gamma = \mathfrak{g}_C / m_{C,\phi}^{(k)},$$

where

$$m_{C,\phi}^{(k)} = \oplus_{\gamma \in C \cap \Gamma, \phi(\gamma) > k} \mathfrak{g}_\gamma$$

is the Lie ideal in $\mathfrak{g}_C$, and $\phi \in \Gamma_{\mathbf{R}}^*$ is the above "cutting" function.

Definition of WCS in pronilpotent case

Then the sheaf of sets $WCS_{\mathfrak{g}_C}$ is defined as the projective limit of the sheaves $WCS_{\mathfrak{g}_{C\phi}^{(k)}}$. One can show that for any open convex subset $U \in \Gamma_{\mathbf{R}}^*$ the set of sections $WCS_{\mathfrak{g}_C}(U)$ admits the following description:

a) For any $y_1, y_2 \in U$ which do not belong to $(\cup_{\gamma \in C \cap (\Gamma - \{0\})} \gamma^\perp) \cap U$ we are given an element $g_{y_1,y_2} \in G_C$ satisfying the cocycle condition.

b) The projections of these elements to $G_{C,\phi}^{(k)}(U)$ satisfy the second condition from the definition of WCS in the nilpotent case (see p.10).

Finally, we generalize this definition by allowing the cone $C \subset \Gamma_{\mathbf{R}}$ to depend on the point $y \in \Gamma_{\mathbf{R}}^*$.

WCS on a topological space

Now we can treat WCS in a vector space as a local model and define WCS in general. For that we need:

1) A Hausdorff locally connected topological space M (then we will speak about WCS on M).

2) A local system of finitely-generated free abelian groups of finite rank $\pi : \underline{\Gamma} \to M$.

3) A local system of $\underline{\Gamma}$-graded Lie algebras $\underline{\mathfrak{g}} = \oplus_{\gamma \in \underline{\Gamma}} \underline{\mathfrak{g}}_\gamma \to M$ over the field $\mathbf{Q}$.

4) A homomorphism of sheaves of abelian groups $Y : \underline{\Gamma} \to \underline{Cont}_M$, where $\underline{Cont}_M$ is the sheaf of real-valued continuous functions on M.

Equivalently we can interpret Y locally as a continuous map from a domain in M to $\Gamma_{\mathbf{R}}^*$. Then we define the pull-back sheaf $WCS_{\underline{\mathfrak{g}}, Y} := Y^*(WCS_{\mathfrak{g}})$, where $WCS_{\mathfrak{g}}$ is the sheaf of sets on $\Gamma_{\mathbf{R}}^*$ constructed in the case of vector spaces.

Definition

A (global) wall-crossing structure on M is a global section of $WCS_{\mathfrak{g},Y}$. The support of WCS σ is a closed subset of $tot(\underline{\Gamma} \otimes \mathbf{R})$ whose fiber over any point $m \in M$ is described such as follows: it is a strict convex closed cone $Supp_{m,\sigma} \subset \underline{\Gamma}_m \otimes \mathbf{R}$ equals to the support of the germ of $WCS_{\underline{\mathfrak{g}},m}$ at the point $Y(m) \in \underline{\Gamma}_m^* \otimes \mathbf{R}$ associated with the section σ.

This definition makes obvious the functoriality of the notion of WCS with respect to pullbacks.

Examples of WCS

1) Let us fix a free abelian group of finite rank Γ together with a Γ-graded Lie algebra $\mathfrak{g} = \oplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$ and a homomorphism of abelian groups $Z : \Gamma \to \mathbf{C}$ (central charge). Then we take $M = \mathbf{R}/2\pi\mathbf{Z}$, and define on M constant local systems with fibers Γ and $\mathfrak{g}$. We set $Y_\theta(\gamma) = Im(e^{-i\theta}Z(\gamma))$, where $\theta \in \mathbf{R}$. Then a WCS associated with this choice is the same as stability data on $\mathfrak{g}$ in the sense of our paper arXiv: 0811.2435.

2) Assume that Γ is endowed with an integer skew-symmetric form $\langle \bullet, \bullet \rangle$. Let us fix central charge Z and set $\mathfrak{g} = \oplus_{\gamma \in \Gamma} \mathbf{Q} \cdot e_\gamma$, where

$$[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2}.$$

We will call it the *torus Lie algebra*. Previous example can be specified to this case. Then the WCS is the same as the collection of numbers $\Omega_Z(\gamma) \in \mathbf{Z}$ (numerical DT-invariants) satisfying our wall-crossing formulas from 0811.2435 (physicists call them KSWCF).

3) The quantum version of the previous example deals with the Lie algebra $\mathfrak{g} = \oplus_{\gamma \in \Gamma} \mathbf{Q}(q^{1/2}) \cdot \widehat{e}_\gamma$ where

$$[\widehat{e}_{\gamma_1}, \widehat{e}_{\gamma_2}] = \frac{q^{\langle \gamma_1, \gamma_2 \rangle / 2} - q^{-\langle \gamma_1, \gamma_2 \rangle / 2}}{q^{1/2} - q^{-1/2}} \widehat{e}_{\gamma_1 + \gamma_2}.$$

Here $\widehat{e}_\gamma = \frac{\widehat{e}_\gamma^{quant}}{q^{1/2} - q^{-1/2}}$ are the normalized generators of the quantum torus $\widehat{e}_{\gamma_1}^{quant} \widehat{e}_{\gamma_2}^{quant} = q^{\langle \gamma_1, \gamma_2 \rangle / 2} \widehat{e}_{\gamma_1 + \gamma_2}^{quant}$. We will call it the *quantum torus Lie algebra*. The corresponding polynomials $\Omega_Z(q^{\pm 1/2}, \gamma) \in \mathbf{Z}[q^{\pm 1/2}]$ are called quantum DT-invariants.

Local picture for WCS: reminder

Recall that locally the WCS is described by a lattice Γ endowed with a skew-symmetric integer form $\langle \bullet, \bullet \rangle : \bigwedge^2 \Gamma \to \mathbf{Z}$ and Γ-graded Lie algebra $\mathfrak{g} = \oplus_{\gamma \in \Gamma} \mathfrak{g}_\gamma$. We impose the condition that if $\langle \gamma_1, \gamma_2 \rangle = 0$ then the corresponding graded components commute. One has

$$\mathfrak{g} = \mathfrak{g}_{\Gamma_0} \oplus \mathfrak{g}_{\Gamma - \Gamma_0}$$

where Γ_0 is the kernel of the skew-symmetric form. The corresponding Lie subalgebra $\mathfrak{g}_{\Gamma_0}$ is central, and it has a complement, which we denoted $\mathfrak{g}_{\Gamma - \Gamma_0}$.

Attractor flow

Observe that $\Gamma_{\mathbf{R}}^*$ is a Poisson manifold foliated by symplectic leaves parallel to the image $i(\Gamma_{\mathbf{R}})$, where $i : \Gamma_{\mathbf{R}} \to \Gamma_{\mathbf{R}}^*$ is defined by the skew-symmetric form. Then in the "global" picture of WCS we will have a Poisson manifold B^0 foliated by symplectic leaves, such that locally the picture looks as in the case of vector spaces. We define *attractor flow* on the subspace of the total space of the local system $\underline{\Gamma} \otimes \mathbf{R} \to B^0$ given by the set of pairs $(b, v), v \in B^0, v \in \underline{\Gamma}_b \otimes \mathbf{R}$ such that $Y(b)(v) = 0$ (here Y is the map from the definition of WCS). Namely, in the local model the flow is defined such as follows:

$$\dot{b} = i(v), \dot{v} = 0.$$

It preserves symplectic leaves and it induces the attractor flow on the subset consisting of (b, γ) where $\gamma \in \underline{\Gamma}_b$ ("integer subset").

Tail set

By definition WCS gives rise to a piecewise constant maps a from the above-mentioned integer subset of $tot(\underline{\Gamma})$, so $(b, \gamma) \mapsto a_b(\gamma) \in \underline{g}_{b,\gamma}$. The discontinuity set of the function a consists of pairs (b, γ) such that γ splits into a sum $\gamma = \gamma_1 + \gamma_2$, where summands are not skew-orthogonal to each other. The attractor flow is transversal to this discontinuity set.

Suppose now that we have a closed subset C^+ in $tot(\Gamma_{\mathbf{R}})$ such that fibers of its projection to B^0 are strict convex cones and which is preserved under the "negative" attractor flow $\dot{b} = -i(v), \dot{v} = 0$.

Then we can define the *tail set* as the set of such pairs (b, γ) from the interior of C^+ that their trajectories under the attractor flow stays in C^+ for all $t > 0$ for which they are defined, and they do not intersect the discontinuity sets of the function a. Typically the restriction of the local system of Lie algebras to the tail set is a local system of rank one.

Initial data for WCS

The *initial data of WCS bounded by* C^+ is defined by the restriction of the function a to the tail set. Then initial data gives rise to a local system of (typically) commutative Lie algebras. In the paper we propose some assumptions which should guarantee that there is a unique WCS with given initial data. The algorithm of this reconstruction is presented in our paper. Roughly, one consider *attractor trees* on B^0 (i.e. trees with edges being attractor flow trajectories, and tail edges come from the tail set) with fixed root $b \in B^0$ and velocity of root edge γ, and apply WCF while moving from the set of tail vertices to $b \in B^0$. In this way we reconstruct $a_b(\gamma)$. The construction is universal. We will see later how it works in the case of complex integrable systems.

C-integrable systems

Let $(X^0, \omega^{2,0})$ be a complex analytic symplectic manifold of complex dimension $2n$.[1] Assume we are given an holomorphic map $\pi : X^0 \to B^0$ such that for any $b \in B^0$ the fiber $\pi^{-1}(b)$ is a complex Lagrangian submanifold of X^0, which is in fact a torsor over an abelian variety endowed with a covariantly constant integer polarization. We will call such data a *(polarized) complex integrable system*. One can generalize this notion to *semipolarized* case (fibers are semiabelian varieties with polarized quotients). From the point of view of variations of Hodge structure the former corresponds to VPHS, while the later to VMHS. Important class of examples : Hitchin systems without singularities and Hitchin systems with singularities.

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[1]Many of the results can be generalized to the case of smooth algebraic varieties.

C-integrable systems with central charge

Let $b \in B^0, \gamma \in \underline{\Gamma}_b := H_1(\pi^{-1}(b), \mathbf{Z})$. Then the map $b \mapsto \int_\gamma \omega^{2,0}$ gives rise to a closed 1-form α_γ. If there exists a global section $Z \in \Gamma(B^0, \underline{\Gamma}^\vee \otimes \mathcal{O}_{B^0})$ such that $d_b Z(\gamma) = \alpha_\gamma$ then Z is called *central charge* of our integrable system. If system admits a central charge then $[\omega^{2,0}] = 0$.

It is expected that Hitchin integrable systems have central charge. Seiberg-Witten integrable systems have central charge. In the case of Hitchin integrable systems without singularities we have polarized integrable systems. If e.g. we have regular singularities, then conjugacy classes of the Higgs field at singular points serve as parameters. Then we have a semipolarized integrable system with central charge. Fixing residues we arrive to a polarized integrable system without central charge.

Pieces of WCS in case of semipolarized systems with central charge

i) We have B^0 which is locally mapped into a vector space $\underline{\Gamma}^*_{b,\mathbf{R}}$ via the map $Y = Im(Z_b)$ (moreover, fixing real θ we can consider $Y_\theta = Im(e^{-i\theta}Z_b)$).

ii) The local system of lattices $\underline{\Gamma}$ carries a skew-symmetric form coming from the semipolarized structure. It can be degenerate, so we have its kernel $\underline{\Gamma}_0 \subset \underline{\Gamma}$ with the symplectic quotient.

iii) Assume that the monodromy of $\underline{\Gamma}_0$ is trivial. Then we have a vector space $Hom(\Gamma_0, \mathbf{C})$, where Γ_0 is a fixed fiber. We can treat B^0 as a family of Kähler manifolds $B^0_{Z_0}$ where $Z_0 \in Hom(\Gamma_0, \mathbf{C})$ (for Hitchin system with R.S. it corresponds to fixing residues at singularities). Each $B^0_{Z_0}$ is the base of polarized integrable system without central charge.

iv) Attractor flow goes along leaves $B^0_{Z_0}$ and is given by gradient lines of the central charge (there are some delicate issues here which we skip).

v) We have the sheaf of Lie algebras $\mathfrak{g} \to B^0$ with fibers being torus Lie algebras associated with $\underline{\Gamma}$ and the skew-symmetric pairing on it.

Full integrable system and the discriminant

Next we are going to discuss the origin of the initial data.
Typically, a complex integrable system $\pi : (X^0, \omega^{2,0}) \to B^0$ arises as an open dense subset in the "full" complex integrable system $\pi : (X, \omega^{2,0}) \to B$. In the latter case there can be degenerate fibers which live over the discriminant set $D := B - B^0$. This is a complex codimension one analytic divisor. We will also assume that there exists an analytic divisor $D^1 \subset D$ such that $dim\, D^1 \leq dim\, B^0 - 2$, the complement $D^0 := D - D^1$ is smooth, and such that our system together with the central charge Z has the following local model near D^0:

A_1-Singularity Assumption

1) There exist local coordinates $(z_1, .., z_n, w_1, ..., w_m)$ near a point of D^0 such that z_1 is small and $D^0 = \{z_1 = 0\}$.

2) The map $Z : B^0 \to \mathbf{C}^{2n+m} \simeq \Gamma^\vee \otimes \mathbf{C}$ is a multi-valued map given in coordinates by

$$(z_1, ..., z_n, w_1, ..., w_m) \mapsto (z_1, ..., z_n, \partial_1 F_0, ..., \partial_n F_0, w_1, ..., w_m),$$

where $\partial_i = \partial/\partial z_i$, and F_0 is given by the formula

$$F_0 = \frac{1}{2\pi i} \frac{z_1^2}{2} \log z_1 + G(z_1, .., z_n, w_1, ..., w_m),$$

and G is a holomorphic function. The Poisson structure on $\mathbf{C}^{2n+m}$ in the standard coordinates $(x_1, ..., x_{2n+m})$ is given by the bivector $\sum_{1 \leq i \leq n} \partial/\partial x_i \wedge \partial/\partial x_{i+n}$.

A_1-Singularity Assumption-continuation

3) The function F_0 (called prepotential) satisfies also a positivity condition coming from the condition $i\langle dZ, \overline{dZ}\rangle > 0$, which is satisfied for the restriction of dZ to symplectic leaves

$S_{c_1,\ldots,c_m} := \{(z_1,\ldots,z_n,w_1,\ldots,w_m)|w_i = c_i\}$.

4) The monodromy of the local system $\underline{\Gamma}$ about D^0 has the form $\mu \mapsto \mu + \langle\mu, \gamma\rangle\gamma$, where γ is such that the pairing $\langle\gamma, \bullet\rangle \in \underline{\Gamma}^\vee$ is a primitive covector.

We expect that the A_1-singularity assumption holds in many realistic examples, e.g. in the case of $GL(r)$ Hitchin integrable systems with singularities.

WCS in the case of semipolarized integrable systems

Let us describe a construction of the WCS. Initial data consists of segments of trajectories of the gradient lines of the functions $b \mapsto |Z_b(\mu)|$ which hit D^0 (the velocity of the trajectory near D^0 is determined by the A_1-singularity assumption) as well as the "tropical" DT invariant $\Omega(b_0, \mu) = 1$ associated with such a trajectory and its point b_0 which is sufficiently close to D^0. Let us fix generic $b \in B^0, \gamma \in \underline{\Gamma}_b$. In order to define WCS we need to assign to this pair the number $\Omega(b, \gamma) \in \mathbf{Z}$ and a strict convex cone $C_b^+ \subset T_b B^0$. I skip the construction of the cone. The "tropical DT-invariant" $\Omega(b, \gamma)$ is defined by induction. We consider all attractor trees (=gradient trees for $|Z_{b'}(\nu)|, \nu \in \underline{\Gamma}_{b'}$) which have b as a root, γ as a root edge (which is straight in the affine structure given by $Y_\theta = Im(e^{-i\theta} Z), \theta = Arg(Z_b(\gamma)))$ and the tail edges hitting D^0, as we have just discussed. It is expected that the number of such trees is finite. Then for every such tree we move from the tail vertices which belong to D^0 toward the root b applying WCF at each internal vertex P. This gives by induction $\Omega(b, \gamma)$ and hence WCS on B^0.

Reminder: wall-crossing formulas

$$\overleftarrow{\prod} T_{\gamma^{out}}^{\Omega(P,\gamma_{out})} = \overrightarrow{\prod} T_{\gamma^{in}}^{\Omega(P,\gamma_{in})},$$

where P is a vertex of the tree and $T_\nu^{\Omega(P,\nu)} : e_\mu \mapsto (1 - e_\nu)^{\Omega(P,\nu)\langle\mu,\nu\rangle} e_\mu$
are symplectomorphisms of the 2-dimensional symplectic subspace in the
tangent space at P corresponding to the edges of the tree outcoming (or
incoming) from P (here e_μ is the generator of the torus Lie algebra) .
Since we know by induction the numbers $\Omega(P, \gamma_{in})$ for outcoming edges,
we can calculate $\Omega(P, \gamma_{out})$ from the wall-crossing formula and proceed
further toward b. Finally, it gives us the desired tropical DT-invariant
$\Omega^{trop}(b, \gamma) := \Omega(b, \gamma)$.

Calabi-Yau 3-folds and Hitchin systems

In our paper with M.K. we describe a "good" class of non-compact Calabi-Yau 3-folds such that the (properly defined) moduli space of good CY 3-folds serves as a base B^0 of a semipolarized complex integrable system. If $b \in B^0$ then the fiber of the local system of lattices $\underline{\Gamma}$ is $\underline{\Gamma}_b = H_3(X_b, \mathbf{Z})/tors$, where X_b is the corresponding 3-fold. One can keep in mind a subclass of examples considered around 2005 by Diaconescu-Donagi-Pantev. They realized all $A - D - E$ Hitchin systems in such a way using families of ALE spaces. The fiber over $b \in B^0$ is the intermediate Jacobian of X_b. In case of the $GL(r)$ Hitchin system on a curve C it is the Jacobian of the corresponding (smooth) spectral curve $S \to C$. Then X_b is a conic bundle over the cotangent bundle T^*C with the spectral curve S_b being the ramification divisor.

As we have seen before, we have the corresponding WCS on B^0 (in fact on $S^1_\theta \times B^0$ when we vary $\theta = Arg(Z_b(\gamma))$) which gives rise to the collection of "tropical" DT-invariants $\Omega(b, \gamma) = \Omega^{trop}(b, \gamma)$ defined in terms of attractor trees.

DT-invariants for non-compact CY 3-folds

On the other hand, under some conditions which I don't have time to discuss now, one can speak about Fukaya categories $\mathcal{F}(X_b)$ endowed with a stability condition depending on $b \in B^0$. The central charge is $Z_b(\gamma) = \int_\gamma \Omega^{3,0}_{X_b}$. The semistable objects are (properly defined) SLAGs on X_b. Such category come with a t-structure generated by spherical stable objects, and hence correspond to quivers with potentials. Thus we can speak about "categorical" DT-invariants $\Omega^{cat}(b, \gamma)$ (virtual number of SLAGs). This story can be interpreted as a special case of WCS described in our Example 2 (stability data on the torus Lie algebra). Varying the point b we arrive to the WCS on B^0 (or on the product of B^0 with S^1). It is natural to expect that the WCS described in terms of Hitchin integrable system is isomorphic to the one described in terms of the Fukaya categories. In particular, we arrive to the following conjecture about DT-invariants:

Conjecture

After the natural embedding of the (universal covering) of the base B^0 of Hitchin integrable system into the space of stability conditions on the Fukaya category of the corresponding non-compact Calabi-Yau 3-fold, we have

$$\Omega^{cat}(b, \gamma) = \Omega^{trop}(b, \gamma).$$

Last year I explained here how this conjecture can be justified in case when the Fukaya category is generated by spherical objects with homology classes given by the velocities of the attractor flow trajectories hitting D^0 (collapsing spherical generators).

Compact CY 3-folds

If X is a compact Calabi-Yau 3-fold then we have a complex integrable system with central charge, but it is neither polarized nor semipolarized. The base $B^0 = \mathcal{L}_X := \mathcal{L} \subset H^3(X, \mathbf{C})$ is the moduli space of deformations of the pair (complex structure on X, holomorphic volume form). The central charge is the same as in the non-compact case: it is given by periods of the holomorphic volume form $\Omega_X^{3,0}$. Then we can speak about attractor flow (same as the gradient flow of the function $F_\gamma(b) = \frac{|Z_b(\gamma)|^2}{vol(X)^2}$). Moreover, because of the compactness of X it is easy to describe a family of strict convex cones such that the velocities of the attractor trajectories belong to the cones. Hence everything looks similar to what we did before except of the one new feature: there exist local minima of the function $F_\gamma(b)$ inside of the moduli space of deformations $\mathcal{M}_X \simeq \mathcal{L}/\mathbf{C}^*$. They are called *attractor points*.

Walls

Let us fix $\gamma \in \Gamma$. Then the corresponding wall is

$$\mathcal{L}_\gamma = \{(\tau, \Omega_\tau^{3,0}) \in \mathcal{L} | \langle Im([\Omega_\tau^{3,0}]), \gamma \rangle = 0, \langle Re([\Omega_\tau^{3,0}]), \gamma \rangle > 0\}.$$

As in the non-compact case, the function $Im(Z)$ endows $\mathcal{L}$ with an integral affine structure. The attractor flow is given by straight lines in this affine structure. The volume function is concave along attractor flow trajectories. The γ-attractor points is described by a point $\Omega^{3,0} \in \mathcal{L}$ such that $Im([\Omega^{3,0}]) = \gamma$. Hence there are countably many attractor points, which we did not have in the case of complex integrable systems (in particular those which correspond to "good" *non-compact CY 3-folds*).

Dynamics of gradient trajectories on $\mathcal{M}_X$ and WCS

The function F_γ descends to a multivalued function on the universal covering of the moduli space of complex structure $\mathcal{M}_X$. Attractor flow on $\mathcal{L}$ projects into the gradient line of this function. Then the gradient trajectory can either hit an (projection of) attractor point or hit the boundary of the natural completion of $\mathcal{M}_X$ (and reaches this boundary in finite time). The boundary points are so-called *conifold points*. They correspond to the discriminant $D = B - B^0$ in our discussion of complex integrable systems (there is also a possibility that the trajectory hits the hypersurface $F_\gamma = 0$ but we are not going to discuss it here). Hence there is an interesting dynamics of the gradient trajectories of F_γ on $\mathcal{M}_X$ related to these possibilities. From the point of view of WCS all that means that the initial data for WCS are described by the tropical DT-invariants Ω^{trop} which are equal to 1 at the smooth locus $D^0 \subset D$ (conifold points) and are arbitrary integers at all attractor points. Thus we have a countably many new parameters (values of DT-invariants at attractor points).

Family of mirror duals in polarized case

Now we return to the WCS for complex integrable systems and review it from the point of view of Mirror Symmetry.

Suppose we have a polarized integrable system $\pi : (X, \omega^{2,0}) \to B$ endowed with a holomorphic Lagrangian section $s : B \to X$. Let us fix $\zeta \in \mathbf{C}^*$ and take $\omega_\zeta = Re(\zeta^{-1}\omega^{2,0})$ as the real symplectic form on X. As the B-field we take $\mathbf{B}_\zeta = Im(\zeta^{-1}\omega^{2,0}) + \mathbf{B}_{can}$, where $\mathbf{B}_{can} \in H^2(X, \mathbf{Z}/2\mathbf{Z})$ is some "canonical" B-field which takes care about some $\pm$ signs in the formulas. Then (under some assumptions) we can construct mirror dual family $X_\zeta^\vee, \zeta \in \mathbf{C}^*$ to the family of wrapped Fukaya categories using our approach from 2004 (SYZ dualization combined with WCF). Each $X_\zeta^\vee$ is a holomorphic symplectic manifold.

Semipolarized case

Let us ignore the parameter ζ for some time.

Suppose that we have a semipolarized system with central charge Z. Then the skew-symmetric form on the homology of fibers has kernel $\underline{\Gamma}_0$. We assume that this local system has trivial monodromy.

Then we have a holomorphic family of complex integrable systems $(X_{Z_0}, \omega_{X_{Z_0}}^{2,0}) \to B_{Z_0}$ parametrized by $Z_0 \in Hom(\Gamma_0, \mathbf{C})$.

Using the same approach as in the polarized case we obtain a holomorphic family of complex symplectic manifolds of mirror duals $X_{Z_0}^\vee := (X_{Z_0}, Re(\omega_{X_{Z_0}}^{2,0}))^\vee$ parametrized by $Hom(\Gamma_0, \mathbf{C})$. The total space of this family will be denoted by $X^\vee$.

In our paper we offer some plausible arguments in favor of the conjecture that it is a pull-back via the map $exp : Hom(\Gamma_0, \mathbf{C}) \to Hom(\Gamma_0, \mathbf{C}^*)$ of the algebraic family of smooth complex symplectic varieties $X^{\vee, alg} \to Hom(\Gamma_0, \mathbf{C}^*)$.

Family over $\mathbf{C}^*$

So far we have been discussing semipolarized integrable systems with fixed holomorphic symplectic form. Let us consider the $\mathbf{C}^*$-family of holomorphic symplectic forms $\omega_\zeta^{2,0} = \omega^{2,0}/\zeta$ on X. Then the corresponding mirror dual Poisson varieties $X_\zeta^{\vee,alg}, \zeta \in \mathbf{C}^*$ form a local system of quasi-affine algebraic varieties over $\mathbf{C}^*$.

Taking the fiber $X_1^{\vee,alg} := X_{\zeta=1}^{\vee,alg}$ we obtain a Poisson variety endowed with a Poisson automorphism $T : X_1^{\vee,alg} \to X_1^{\vee,alg}$, which is equal to id on the algebra of central functions (the latter is isomorphic to $\mathcal{O}(Hom(\Gamma_0, \mathbf{C}^*)))$. Similar automorphisms appear in the theory of cluster algebras (Coxeter automorphisms).

WCS from the point of view of Mirror Symmetry

In the case of semipolarized integrable systems with central charge and holomorphic Lagrangian section we have Kähler metrics on the bases of the corresponding polarized integrable systems. The edges of the gradient trees of the function $|Z_b(\gamma)|$ are straight segments in the dual **Z**-affine structure. In terms of the central charge, the dual affine structure for the symplectic form $Re(\omega^{2,0}/\zeta), |\zeta| = 1$ is given by $Y_\theta := Im(e^{-i\theta}Z)$ with fixed restriction of Y_θ to Γ_0, where $\zeta = e^{i\theta} \in \mathbf{C}^*$.

The SYZ approach to Mirror Symmetry gives rise to an inductive procedure of constructing walls and changes of coordinates, starting with certain data assigned to generic points of the discriminant $B - B^0$. Namely, for a point $b \in B^0$ which is sufficiently close to a generic point of the discriminant, one counts limiting pseudo-holomorphic discs whose projection to the base is a short gradient segment connecting the point b with a point of $B - B^0$. This inductive procedure is a priori different from the one with attractor trees.

MS gives an alternative point of view on WCS

Nevertheless one can prove by induction (moving along the oriented gradient tree from the discriminant to a given point) that the walls and the changes of coordinates in Mirror Symmetry story coincide with those in the attractor trees story. This can be thought of as an alternative approach to the construction of WCS. For example, the initial data for which $\Omega(\gamma) = 1$ for A_1-singularities correspond to the count of pseudo-holomorphic discs in the standard A_1-singularity model

Family of mirror duals

Let us consider polarized case for simplicity.

Suppose we are given a complex integrable system $\pi : (X, \omega^{2,0}) \to B$ endowed with a holomorphic Lagrangian section $s : B \to X$. We will assume that the fibers of π are abelian varieties, but do not assume that they are polarized. Let $\underline{\Gamma} \to B^0$ be the corresponding local system of lattices over the complement to the discriminant. Suppose we are given a class $\beta \in H^1(B^0, \underline{\Gamma}^\vee \otimes (\mathbf{R}/2\pi\mathbf{Z}))$ which comes from the class $\beta_X \in H^2(X, \mathbf{R}/2\pi\mathbf{Z})$ which vanishes on $s(B)$ and on fibers of π. Let us consider the holomorphic family of the Fukaya categories $\mathcal{F}(X, Re(\omega^{2,0}/\zeta), \mathbf{B} = Im(\omega^{2,0}/\zeta) + \beta_X)$. Then mirror duals $X_\zeta^\vee, \zeta \in \mathbf{C}^*$ form a holomorphic family of algebraic varieties over $\mathbf{C}$ (which are symplectic and also expected to be defined over $\mathbf{Z}$ in the polarized case).

Dual integrable systems

Definition

Dual integrable system is a complex integrable system $Y \to B$ such that its restriction to B^0 is obtained by:
a) taking dual abelian varieties to fibers of π;
b) replacing a) by the torsor corresponding to β.

Notice that there is a holomorphic Lagrangian section $B^0 \to Y$ of the dual integrable system. Let us assume that it extends to the section $B \to Y$.

Conjecture

The above family $X_\zeta^\vee$ of mirror duals extends to $\zeta = 0$ holomorphically in such a way that the fiber at $\zeta = 0$ is isomorphic to the dual integrable system.

There is a similar story (and an analogous conjecture) in the semipolarized case.

Skeleta and Hitchin systems

Two conjectures below are formulated for simplicity in the polarized case. There are versions of them in the case of semipolarized integrable systems with central charge and holomorphic Lagrangian section. In the second conjecture we use the notion of skeleton of a log Calabi-Yau. This notion is similar to the notion of the skeleton of maximally degenerate CY introduced in our 2004 paper. It is a $\mathbf{Z}PL$-space, which can be defined in terms of the CW complexes associated with divisors of s.n.c. compactifications on which the holomorphic volume form has poles of order 1. In the case of e.g. $SL(2)$ Hitchin system such a compactification is related to the Stokes phenomenon for the corresponding ζ-connection. We conjecture that in general the Betti realization of the Hitchin integrable system is isomorphic to the mirror dual $X_\zeta^\vee, \zeta \in \mathbf{C}^*$ of the (ζ-rescaled) Dolbeault realization. These spaces are expected to be log CYs with some nice properties (which I don't have time to discuss). Moreover, the corresponding skeleta should be isomorphic to the bases of those Hitchin integrable systems.

Conjectures about extension at $\zeta = 0$

Conjecture

Let us fix a point $b \in B^0$ in the base of a complex integrable system
$\pi : X^0 \to B^0$ with abelian fibres endowed with a complex Lagrangian
section $B^0 \to X^0$. Let us fix a point $e^{i\theta} \in S^1$ such that the pair $(e^{i\theta}, b)$
does not belong to the wall in $M = S^1 \times B^0$. Then the constant family of
complex symplectic manifolds $X^{\vee}_{te^{i\theta}}$ over an open ray $l_\theta = \mathbf{R}_{>0}e^{i\theta}$ can be
extended to a C^∞ family over the closed ray $\mathbf{R}_{\geq 0}e^{i\theta}$ in such a way that
the fiber at $t = 0$ is a real integrable system over Sk_θ. Here Sk_θ is the
skeleton of $(X^0)^{\vee}_{e^{i\theta}}$.

Conjecture

For any $e^{i\theta} \in S^1$ the corresponding Sk_θ is **Z**PL-manifold isomorphic to B
which is endowed with the affine structure derived from the symplectic
form $Re(e^{-i\theta}\omega^{2,0})$ on X^0.

MS story: volume preserving transformations

In the case of SYZ picture of Mirror Symmetry the construction of mirror dual involves transformations which locally preserve the volume form rather than a Poisson structure. In this case $\mathfrak{g}$ is the Lie algebra of divergence free vector fields on $Hom(\Gamma, \mathbf{C}^*)$ and there is no distinguished skew-symmetric form on Γ. The lattice Γ is Γ_b, the first homology group of a fiber of a *real* integrable system at a given point $b \in B^0$. Differently from the case of symplectomorphisms when the dimension of the graded component is equal to 1, we now have $\dim \mathfrak{g}_\gamma = n - 1$, where $n = rk\,\Gamma$ for $\gamma \neq 0$.

Explicit formulas for the Lie bracket

Explicitly, the Lie algebra of vector fields on the algebraic torus $Hom(\Gamma, \mathbf{C}^*)$ is spanned by elements $x^\gamma \partial_\mu$ where $\gamma \in \Gamma$, $\mu \in \Gamma^\vee$ satisfying the linear relations

$$x^\gamma \partial_{\mu_1} + x^\gamma \partial_{\mu_2} = x^\gamma \partial_{\mu_1 + \mu_2} .$$

Derivation ∂_μ is a constant vector field in logarithmic coordinates. The commutator rule is given by

$$[x^{\gamma_1} \partial_{\mu_1}, x^{\gamma_2} \partial_{\mu_2}] = x^{\gamma_1 + \gamma_2} \big((\mu_1, \gamma_2) \partial_{\mu_2} - (\mu_2, \gamma_1) \partial_{\mu_1} \big) .$$

The subalgebra $\mathfrak{g}$ of divergence-free vector fields is spanned by elements $x^\gamma \partial_\mu$ with $(\mu, \gamma) = 0$. It is obviously graded by lattice Γ. Similarly to the symplectic (and also Poisson) case, the graded complement to $\mathfrak{g}_0$ is a Lie subalgebra $\mathfrak{g}' = \oplus_{\gamma \neq 0} \mathfrak{g}_\gamma$ in $\mathfrak{g}$ (notice that an analogous property *does not* hold for the Lie algebra of all vector fields).

Pair of lattices

One can generalize the above considerations to the following situation. Suppose we are given two lattices Γ_1, Γ_2 and an integer pairing between them $(\bullet, \bullet) : \Gamma_2 \otimes \Gamma_1 \to \mathbf{Z}$. We do not assume that the pairing is non-degenerate. We denote by $\Gamma_{1,0} \subset \Gamma_1$ and $\Gamma_{2,0} \subset \Gamma_2$ the corresponding kernels.

Then we consider the Lie algebra $\mathfrak{g} := \mathfrak{g}_{\Gamma_1, \Gamma_2, (\bullet, \bullet)}$ spanned by elements $x^\gamma \partial_\mu$ where $\gamma \in \Gamma_1$ and $\mu \in \Gamma_2$ such that $(\mu, \gamma) = 0$, satisfying the same relations as above. It contains the Lie subalgebra

$$\mathfrak{g}' := \oplus_{\gamma \in \Gamma_1 - \Gamma_{1,0}} \mathfrak{g}_\gamma.$$

The previous special case corresponds to $\Gamma_1 = \Gamma$, $\Gamma_2 = \Gamma^\vee$. In general, $\mathfrak{g}$ can be thought as the Lie algebra of divergence-free vector fields on a torus, preserving a collection of coordinates and commuting with a subtorus action.

Analog of WCS in a vector space-1

Now we are ready to describe the analog of a WCS for Lie algebra $\mathfrak{g}'$.
The main difference with the previous is that now walls are hyperplanes in
$\Gamma^*_{2,\mathbf{R}} := \Gamma^\vee_2 \otimes \mathbf{R}$ (and not in the dual space to the grading lattice Γ_1). We
define a wall as a hyperplane in $\Gamma^*_{2,\mathbf{R}}$ given by $\mu^\perp$, where $\mu \in \Gamma_2 - \Gamma_{2,0}$
(one may assume that μ is primitive). With any wall $H \subset \Gamma^*_{2,\mathbf{R}}$ we
associate a graded Lie subalgebra

$$\mathfrak{g}_H := \bigoplus_{\gamma \in \Gamma_1} \mathfrak{g}_{H,\gamma} \subset \mathfrak{g}'$$

spanned by $x^\gamma \partial_\mu$ such that $(\mu, \gamma) = 0$ and $\gamma \in \Gamma_1 - \Gamma_{1,0}$. As in the Poisson
case, this Lie algebra is abelian. It is convenient to associate with any γ as
above a nonzero constant vector field on the hyperplane H equal to
$\iota(\gamma) := (\bullet, \gamma) \in \Gamma^\vee_2 \subset \Gamma^*_{2,\mathbf{R}}$. In SYZ picture the trajectories of this vector
field are (possible) parts of tropical trees corresponding to analytic discs
with the boundary on a small Lagrangian torus, the fiber of SYZ fibration.

Analog of WCS in a vector space-2

Also with any **Q**-vector subspace $V \subset \Gamma^*_{2,\mathbf{R}}$ which is the intersection of two walls we associate a graded Lie algebra $\mathfrak{g}_V$ (which is a *not* a subalgebra of $\mathfrak{g}'$) such as follows. As a Γ_1-graded vector space $\mathfrak{g}_V$ will be equal to the direct sum $\oplus_{H \supset V} \mathfrak{g}_H$ over all walls containing V. The Lie bracket on $\mathfrak{g}_V$ is defined as follows. Let $(x^\gamma \partial_\mu)_H$ where $\gamma \in \Gamma_1 - \Gamma_{1,0}, \mu \in \Gamma_2 - \Gamma_{2,0}$ denotes the element $x^\gamma \partial_\mu \in \mathfrak{g}_H$ considered as an element of $\mathfrak{g}_H \subset \mathfrak{g}_V$, where $H = \mu^\perp$ is a wall containing V. Then we define the Lie bracket by the formula:

$$[(x^{\gamma_1} \partial_{\mu_1})_{H_1}, (x^{\gamma_2} \partial_{\mu_2})_{H_2}] = (x^{\gamma_3} \partial_{\mu_3})_{H_3},$$

in case if $H_i = \mu_i^\perp, i = 1, 2, 3$, $\gamma_3 = \gamma_1 + \gamma_2$, $\mu_3 = (\mu_1, \gamma_2)\mu_2 - (\mu_2, \gamma_1)\mu_1$ and $\mu_3 \notin \Gamma_{2,0}$. Otherwise. i.e. if $\mu_3 \in \Gamma_{2,0}$ (and as one can easily see $\mu_3 = 0$) we define the commutator to be equal to zero.

Analog of WCS in a vector space-3

The we consider the pronilpotent case by choosing a strict convex cone $C \subset \Gamma_1 \otimes \mathbf{R}$, and working with $\mathfrak{g}_C := \prod_{\gamma \in \Gamma \cap C - \Gamma_{1,0}} \mathfrak{g}_\gamma$. Then for a given functional $\phi : \Gamma_1 \to \mathbf{Z}$ which is nonnegative and proper on the closure of C, we consider finite-dimensional nilpotent quotients

$$\mathfrak{g}_{C,\phi}^{(k)} = \oplus_{\gamma \in \Gamma_1 - \Gamma_{1,0} | \phi(\gamma) \leq k} \mathfrak{g}_{C,\gamma} = \mathfrak{g}_C / m_{C,\phi}^{(k)}$$

where $m_{C,\phi}^{(k)} := \prod_{\gamma \in \Gamma_1 - \Gamma_{1,0} : \phi(\gamma) > k} \mathfrak{g}_{C,\gamma}$ is an ideal in $\mathfrak{g}_C$.

Similarly we define the Lie algebras $\mathfrak{g}_{H,C,\phi}^{(k)}$ and $\mathfrak{g}_{V,C,\phi}^{(k)}$.

Let us fix finitely many walls $H_i, i \in I$. We define the set $WCS_k(\{H_i\}_{i \in I}, C, \phi)$ of wall-crossing structures for $\mathfrak{g}_{C,\phi}^{(k)}$ which are supported on the union $\cup_{i \in I} H_i$ in the following way. First we observe that the walls $H_i, i \in I$ give rise to the natural stratification of $\Gamma_{2,\mathbf{R}}^*$. Then an element of $WCS_k(\{H_i\}_{i \in I}, C, \phi)$ is a map which associates an element g_τ of the group $\exp(\mathfrak{g}_{H_i,C,\phi}^{(k)}), i \in I$, where $\tau \subset H_i$ is a co-oriented stratum of codimension one in $\Gamma_{2,\mathbf{R}}^*$ (notice that τ is an open subset of H_i).

Analog of WCS in a vector space-4

The only condition on this map says that for any generic closed loop $f : \mathbf{R}/\mathbf{Z} \to \Gamma_{2,\mathbf{R}}^*$ surrounding a codimension two stratum $\rho \subset V, codim_{\mathbf{R}} V = 2$, the product of images of the corresponding elements $exp(g_{\tau_{t_i}})$ in $exp(\mathfrak{g}_{V,C,\phi}^{(k)})$ over the finite sequence of intersection points $f(t_i)$ of the loop with walls H_i is equal to the identity.

Now we take the inductive limit of the sets $WCS_k(\{H_i\}_{i \in I}, C, \phi)$ over all finite collections $\{H_i\}_{i \in I}$ of walls and after that we take the projective limit over k. The resulting set $WCS_{\mathfrak{g},C}$ is our analog of WCS relevant to the Mirror Symmetry.

There is an analog of the initial data for WCS in this framework. I skip this discussion.

Analog of WCS associated with SYZ approach to MS-1

I summarize main points.

1) In 2000 we proposed two main approaches to MS: via Gromov-Hausdorff collapse and via non-archimedean geometry. In the first one we have a maximally degenerate family $X_t, t \to 0$ of complex CY manifolds. Conjecturally (true for abelian varieties, checked by Gross-Wilson for K3) X_t converges in GH sense to the base B of a real integrable system, which carries on a dense open subset $B^0 \subset B$ a **Z**-affine structure and a Riemannian metric which satisfies real Monge-Ampère equation. For any point $b \in B^0$ we can define the lattice $\Gamma_{1,b} := H_2(X_t, \pi_t^{-1}(b), \mathbf{Z})$ (the latter stabilizes as $t \to 0$ along a ray). This family of lattice gives rise to a local system $\underline{\Gamma}_1 \to B^0$. It can be extended to a local system on $S_t^1 \times B^0$. We conjectured (and assume here) that $codim(B - B^0) \geq 2$. Moreover we assume that the affine structure has A_k singularity.

Analog of WCS associated with SYZ approach to MS-2

There is a natural projection $p : \underline{\Gamma}_1 \to T^{\mathbf{Z}}$, where $T^{\mathbf{Z}} := T^{\mathbf{Z}}_{B^0} \subset T_{B^0}$ is the locally covariant lattice which defines the $\mathbf{Z}$-affine structure on B^0. Denoting $\underline{\Gamma}_{1,0} = Ker(p)$ we obtain an exact sequence of lattices

$$0 \to \underline{\Gamma}_{1,0} \to \underline{\Gamma}_1 \to T^{\mathbf{Z}}.$$

One can derive a similar geometry from non-archimedean approach as well. In what follows we will assume that the local system $\underline{\Gamma}_{1,0}$ is trivial (this condition is automatically satisfied in most of examples).

Analog of WCS associated with SYZ approach to MS-3

Then everything looks very similar to the case of complex integrable systems, but there are some important issues which are handled differently. In particular, there is a notion of *non-negative* $(1,1)$-*currents* on B. It can be thought of as a limit of Kähler metrics on X_t or spelled entirely in non-archimedean terms (work of Boucksom-Favre-Jonson). Using this notion we formulate conditions which are sufficient for the existence of cones $C_b, b \in B^0$ where the velocities of the gradient trajectories live (the latter correspond to attractor trajectories and come from limits of of pseudo-holomorphic discs). Furthermore considering non-negative currents which are GH-limits of ample effective divisors $S_t \subset X_t$ we can ensure the finiteness of the number of gradient trees on B with the fixed generic root $b \in B^0$ and fixed direction γ of the root edge. Such currents ("tropical effective divisors") are related to "slabs" in the Gross-Siebert story (which produces in the "toric" framework a WCS in the above sense).

Having a WCS in this sense we use the same machinery as in the case of complex integrable systems and obtain a CY manifold over a non-archimedean field $\mathbf{C}((t))$.

Summary

We have discussed the notion of WCS in the framework of DT-theory, complex integral system with central charge and MS. Most of what I have said is based on non-trivial geometric assumptions, which we can check in some special cases (or present plausible arguments in favor of them like e.g. existence of the well-defined count of SLAGs on our non-compact CY 3-folds). Besides of the obvious open problems of proving those assumptions, there are some other interesting questions, e.g. the relationship with quantized complex integrable systems (e.g. with work of Nekrasov-Shatashvili) or the relationship with cluster varieties. And many others...