M. Kontsevich - 30/1/13 - Integrality for DT series

<u>Recall:</u> WCF on 3-CY cat. + stability $\rightsquigarrow$ DT-invariants

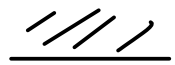
A huge class of examples comes from quivers w/ potentials, $(Q, W \in \widehat{\mathbb{C}Q}/[\,,\,])$

DT gives a formal diffeo $\varphi : \mathbb{Z}[[x_1 \dots x_N]] \circlearrowleft$ $\quad N = \#\,\mathrm{vertices}(Q)$

$$x_i \mapsto x_i + \dots$$

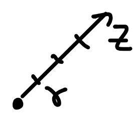
Poisson bracket $\{x_i, x_j\} = a_{ij}\, x_i x_j$, $\quad a_{ij} = \#\, \overset{i}{\bullet} \to \overset{j}{\circ} \; - \; \#\, \overset{j}{\bullet} \to \overset{i}{\bullet}$

Fix any generic stability cond., $Z : \mathbb{Z}^N \to \mathbb{C}$, $\gamma \mapsto \sum z_i \gamma^i$
$\gamma = (\gamma^i)$ $\qquad \mathrm{Im}\, z_i > 0$.

Then $\mathbb{Z}^N_{\geq 0} \longrightarrow \overline{\mathbb{H}} =$ 

$\Rightarrow \exists!$ Decomposition $\displaystyle \varphi = \prod_{\substack{\gamma \in \mathbb{Z}^N_{\geq 0} - \{0\} \\ \mathrm{Arg}\, Z(\gamma) \searrow}} T_\gamma^{\Omega_Z(\gamma)}$

where $T_\gamma : x_i \mapsto (1 \pm x^\gamma)^{\sum a_{ij} \gamma^j} x_i$

$\Rightarrow$ for generic Z, for any ray  get a series in $\underline{1}$ variable x^γ

$$x^\mu \mapsto \varphi_\gamma(x^\gamma)^{\langle \gamma, \mu \rangle} x^\mu \quad \text{where } \varphi_\gamma = 1 + \dots \in \mathbb{Z}[[x^\gamma]]$$

Can try to study these series & their properties.
Hopeless in all generality, but many interesting examples.

<u>Ex:</u> Kronecker quiver $1 \rightrightarrows 2$ (k arrows) $\quad \Rightarrow \varphi = T_{1,0}^k \circ T_{0,1}^k$

$T_{1,0} : (x_1, x_2) \mapsto (x_1(1 - x_2), x_2)$

$T_{0,1} : (x_1, x_2) \mapsto (x_1, (1 - x_1) x_2)$

$\rightsquigarrow$ for $k \geq 3$:

(for $k = 1, 2$ it's tamer)

<u>Claim:</u> all φ_γ are <u>algebraic</u>!

- For a large class of quivers Q w/ generic potentials, φ is <u>rational</u>.
  [$\leftrightarrow$ cluster story...]

- $\forall$ quiver Q with 0 potential, what is φ?

$$x_i = \frac{1-y_i}{\prod y_j^{b_{ij}}} \quad \text{determines} \quad y_i = 1 - x_i + \dots \in \mathbb{Z}[[x_1 \dots x_N]] \quad \text{algebraic.}$$

now $$\tilde{x}_i = \frac{1-y_i}{\prod y_j^{b_{ji}}} \quad ; \quad \varphi : x_i \mapsto \tilde{x}_i .$$

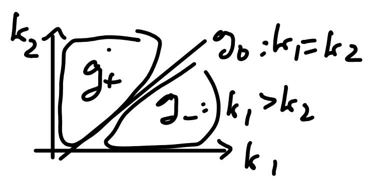
<u>Preservation of algebraicity:</u>

$\varphi : \mathbb{C}[[x_1, x_2]] \circlearrowleft$ preserving $\omega = \frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2}$, $x_i \mapsto x_i + \dots$

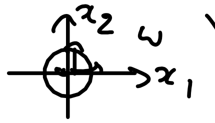
$\to \exists!$ decomp. $\varphi = \varphi_- \cdot \varphi_0 \cdot \varphi_+$

$$\begin{array}{ccccccc} \varphi & = & \varphi_- & \cdot & \varphi_0 & \cdot & \varphi_+ \\ \cap & & \cap & & \cap & & \cap \\ G & = & G_- & \cdot & G_0 & \cdot & G_+ \end{array}$$

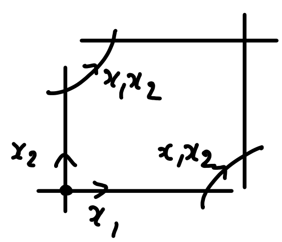
where $\mathfrak{g} = \text{Lie } G = \left\{ x_1^{k_1} x_2^{k_2} \,/\, (k_1, k_2) \geq 0, \neq (0,0) \right\}$

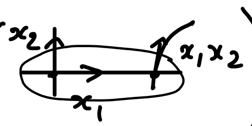
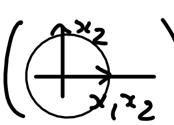


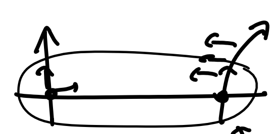
<u>Thm:</u> φ is algebraic $\iff$ $\varphi_+, \varphi_0, \varphi_-$ are algebraic

<u>Proof</u> $\Rightarrow$: $G = \text{Aut}\Big($  $\Big)$ formal scheme with Poisson $\{.,.\}$ and 2 fixed tangent vectors

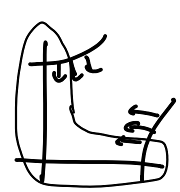
Consider $\mathbb{P}^1 \times \mathbb{P}^1$ & blowup at $(0, \infty)$ and $(\infty, 0)$:
$\quad\quad\quad\quad x_1 \quad x_2$



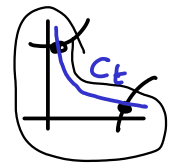
Then $\text{Aut}\Big($  $\Big) = \text{Aut}\Big($  $\Big) = G_- . G_0$.

$G_- = \text{Aut}\Big($  $\Big)$
w/ trivi of normal bundle to this $\mathbb{P}^1$

$\rightsquigarrow$ $G_- \backslash G / G_+$ = Moduli space of formal schemes
$\quad \cong G_0$

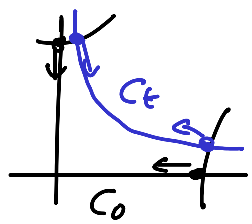


Now consider pts close to origin on the formal $\mathbb{P}^1$'s



formal nbd.

deform. theory $\Rightarrow$ complement to coord. axes is filled by rational curves $C_t \simeq \mathbb{P}^1$, $t = x_1 x_2$ with 2 marked pts & marked tangent vectors



C_0 deforms to C_t

Cross-ratio of these on $C_t \rightsquigarrow$ f$^{\underline{n}}$ of $t = x_1 x_2$: <u>this is φ_0</u>.

Hence φ alg. $\Rightarrow$ φ_0 algebraic.

Similarly for φ_+, φ_-. ∎

φ preserves $\frac{dx_1}{x_1} \wedge \frac{dx_2}{x_2} \rightsquigarrow K_2$ (field of rat'l functions) $\to$ closed 2-form

for a field, $K_2(F) = \Lambda^2 K_1(F) / [f] \wedge [1-f]$

where $K_1(F) = F^*$

$\rightsquigarrow$ get a K_2-symplectomorphism.

Graphs of such are K_2-Lagrangians.

i.e. $\underset{p_i, q_i}{(\mathbb{C}^*)^{2n}} \supset L$, $\sum p_i \wedge q_i|_L = 0$.

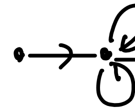
Relation to ... $\partial M^3 = \Sigma^2 \Rightarrow \mathrm{Rep}(\pi_1(\Sigma^2), G) \simeq (\mathbb{C}^*)^{2n}$
$\cup$
$\mathrm{Rep}(\pi_1 M, G)$

<u>Example</u>: $f(t) = \dfrac{\sqrt{1+4t} - 1}{2t} = 1 - t + 2t^2 - 5t^3 + \cdots$ generating series of Catalan #'s

$f = 1 - t f^2$

then $[f] \wedge [t] = 0 \in K_2$(alg. curve).

Write $f(t) = \prod_{n \geq 1} (1 - t^n)^{c(n)}$: <u>claim</u> $\Omega(n) = \dfrac{c(n)}{n} \in \mathbb{Z}$!

In fact, $\Omega(n)$ is DT invt of some quiver:  (zero potential)

$$\left(\text{combinatorially, } \Omega(n) = \#\left\{ A \subset \mathbb{Z}/2n\mathbb{Z} \;\middle|\; \begin{array}{l} \#A = n \\ \sum_{a\in A} a = 1 \bmod n \end{array} \right\} \Big/ \mathbb{Z}/2n\mathbb{Z} \right)$$

- 1st proof of integrality of $\Omega(n)$:

$$[f]\wedge[t] = \prod (1-t^n)^{c(n)} \wedge [t] = \sum_{n\geq 1} \frac{c(n)}{n} [1-t^n]\wedge[t^n]$$

vanishes in $K_2\left(\mathbb{Z}[[t]]/t^M\right) = \prod_{n=1}^{M} \mathbb{Z}/n\mathbb{Z}$ $\Rightarrow$ hence $c(n) \equiv 0 \bmod n$. ∎

- 2nd ....