M. Kontsevich – 29/1/13 – Weak CY algebras (w/ Vlassopoulos)

Compact case ("classical")

A A_∞-alg./$\mathbb{C}$, $\dim A < \infty$ $\quad A \in \mathrm{Perf}(\mathbb{C}\text{-mod})$

Def: CY structure $\Leftrightarrow$ cyclic def. structure $\quad (\cdot,\cdot) : A^{\otimes 2} \to \mathbb{C}$

$\Leftrightarrow$ solution of MC eqn in necklace Lie algebra

$\prod (A^{\otimes n})/_{\mathbb{Z}/n\mathbb{Z}}$, with $[\times, \succ\!\!-] = \Sigma$ … pair by $(\cdot,\cdot)$

Classification: (K.-Soibelman) : $HH_\bullet(A) \to HC_\bullet(A) \xrightarrow[\alpha]{} \mathbb{C}$

$(n \geq 3)$

$$(HH_\bullet(A))^* = R\mathrm{Hom}_{A\otimes A^{op}\text{-mod}}(A, A^*)$$

st. Nondegeneracy: The class $\alpha \in HC_\bullet(A)^*$ gives us iso. $A[d] \simeq A^*$

Statement: CY structures on an A_∞-alg. A $\underset{1:1}{\Longleftrightarrow}$ nondegenerate $\alpha \in HC_\bullet(A)^*$

PROP action for CY alg. = structures on $H := HH_\bullet(A, A)$:

$$H_\bullet(M_{g,\vec{n}_1,\vec{n}_2}) \otimes H^{\otimes n_1} \to H^{\otimes n_2} \qquad \forall g,\ n_1 \geq 1,\ n_2 \geq 0$$

n_1 framed punctures $\quad$ n_2 framed punctures

Smooth (non compact) case : **smooth** $:= A \in \mathrm{Perf}(A \otimes A^{op}\text{-mod})$

Let $A^\vee := \mathrm{Hom}_{A\otimes A^{op}\text{-mod}}(A, A\otimes A^{op})$

Classification (K–Vlassopoulos): CY structure $:=$ class $\beta \in HC^-(A)$ negative cyclic homology

st. natural image of β under $HC^-(A) \xrightarrow[\text{quotient}]{} HH_\bullet(A,A) = R\mathrm{Hom}_{A\otimes A^{op}}(A^\vee, A)$

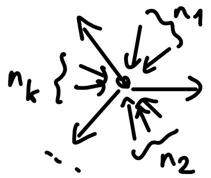
is a quasi iso. $A[-d] = A^\vee$.

PROP action: $H_\bullet(M_{g,\vec{n}_1,\vec{n}_2}) \otimes H^{\otimes n_1} \to H^{\otimes n_2} \qquad \forall g,\ n_1 \geq \underline{\underline{0}},\ n_2 \geq \underline{\underline{1}}.$

Goal: generalization that includes both compact & smooth cases.

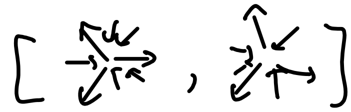
Def: Weak CY algebra A (no finiteness assumption) over k, $\mathrm{char}(k)=0$
$\mathbb{Z}$ or $\mathbb{Z}/2$-graded,
structure maps $m_{n_1,\dots n_k} : A^{\otimes n_1} \otimes \dots \otimes A^{\otimes n_k} \to A^{\otimes k}$
cyclically invariant $\quad \forall k \geq 1,\ n_1 \dots n_k \geq 0$ except $m_0 = 0$.

Interpret as :



"directed necklace Lie alg"

Solves Maurer-Cartan eq$^{\underline{n}}$ where



$= \sum$ ways of pairing an input arrow w/ an output arrow.

The $k=1$ part of the structure $\iff$ A_∞-structure

Higher Hochschild complex for A_∞-alg:

$$C^{(k)}(A,A) := R\mathrm{Hom}_{A^{\otimes k} \otimes A^{\otimes k\, op}\text{-mod}}\left(A^{\otimes k}, \sigma_k A^{\otimes k}\right)$$

where $\sigma_k = (1\,2 \dots k)$ cyclic permutation

Then $\prod C^{(k)}/_{\mathbb{Z}/k}$ is naturally a Lie algebra, q.iso to the directed necklace Lie alg.

Observe $HH^{(k)}_\bullet(A,A) = \begin{cases} R\mathrm{Hom}_{\mathrm{Fun}(A,A)}(\mathrm{Id}, S^{1-k}) & \text{Serre functor; makes sense for smooth: } S^{-1} := A^\vee \otimes - \\ R\mathrm{Hom}_{\mathrm{Fun}(A,A)}(S^{k-1}, \mathrm{Id}) & \text{makes sense for compact } S := A^* \otimes_A - \end{cases}$

* The notion is derived Morita invt. $\Rightarrow$ can choose generators ...

<u>Examples of weak CY algs.:</u>

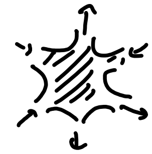
- X smooth scheme $/\mathbb{C}$, $s \in \Gamma(X, K_X^{-1})$ $\rightsquigarrow$ weak CY str. on $A \simeq D^b(X)$
  (A = End of some compact generator of $D^b(X)$)

namely, s gives the "first order" correction (case $k=2$) $\in HH_\bullet^{(2)}(A)$

(HKR-type result: $HH^{(k)}(A,A) \simeq R\mathrm{Hom}_{X\times X}(\mathcal{O}_\Delta, K_{X,\Delta}^{\otimes 1-k})$)

and then higher order corrections can be zero

- (?) mirror dual: $FS\begin{pmatrix} Y \\ \downarrow w \\ \mathbb{C} \end{pmatrix}$ Fukaya-Seidel cat. should be weak CY.

(structure maps  should count pseudohol. discs 

with ∂ on given generators [thimbles]
and w/ incoming & outgoing ends).

- M finite CW-complex, conn., m_0 base point

$A :=$ chains on $\Omega(M, m_0)$ (topological monoid)
is homologically smooth

Any class $\beta \in H_*(M)$ induces a weak CY structure

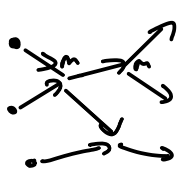
while (M,β) w/ Poincaré duality as CY structure

- Seidel: M mfld w/ ∂, then $H_*(M, \partial M)$ is weak CY.

Graphical calculus: $H_\bullet(M_{g,\vec{n}_1,\vec{n}_2}) \otimes H^{\otimes n_1} \to H^{\otimes n_2}$ $\quad H = HH_\bullet(A,A)$

$n_1, n_2 \geq 1$

Use acyclic oriented ribbon graphs to define these



3 types of vertices:

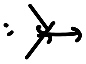
- n_1 sources $\bullet\!\!\to$ $\quad \deg_{in} = 0$, $\deg_{out} = 1$

encodes operation

- inner vertices: $\deg_{out} \geq 1$
and if $\deg_{out} = 1$ then $\deg_{in} \geq 2$.

i.e. simplest two are

- n_2 output vertices $\to\!\!\bullet$ $\quad \deg_{in} = 1$, out $= 0$

$m_{0,0}$:  $A^0 \to A^{\otimes 2}$, m_2:  $A^{\otimes 2} \to A$

given a ribbon graph, place operations m at inner vertices & get operation $H^{\otimes n_1} \to H^{\otimes n_2}$ induced by that cell in $M_{g,\vec{n}_1,\vec{n}_2}$.

Graphs are oriented-acyclic (no directed cycles) since have "height" fn along our genus g surface (but still has $b_1 = g$)

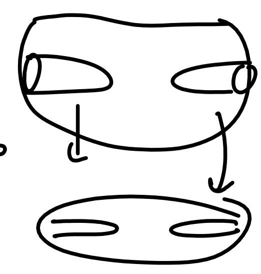
(Model: metrized unoriented ribbon graphs with n_1 ∞ legs
n_2 numbered vertices
all other vertices deg ≥ 3
+ "height" fn $h: \Gamma \to \mathbb{R}$ ....)

- Applications: algebrization of "string topology" by applying this to $C_*(\Omega M)$

- Also should have: for LG model $\begin{array}{c} Y \\ \downarrow w \\ \mathbb{C} \end{array}$ ~ proper, $H = H_\bullet(Y, w^{-1}(+\infty))$
$H^\vee = H_\bullet(Y, w^{-1}(-\infty))$

operations count holom. curves with punctures $\to$ cycles

inputs = $(-\infty)$-thimbles



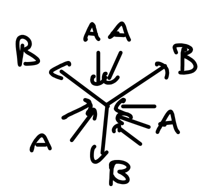
outputs = $(+\infty)$-thimbles

(claim: these holom. curves live entirely in fiber by considering proj to $\mathbb{C}$)
$\to$ GW(fiber)
(???)

(Mohammed says: want punctures $\to \infty$ less restricted
& then count genus g holom. multisections of w)

- Weak CY algs form a category

morphism of weak CY alg. $A \to B$ = collection of maps $A^{\otimes m_1} \otimes \cdots \otimes A^{\otimes n_k} \to B^{\otimes k}$
$n_i \geq 1$, $k \geq 1$.



<u>Origin:</u> weak CY is a noncommutative generalization of $\underset{\text{Lagr.}}{L} \subset \underset{\text{symplectic}}{X}$

$X =$ formal nbd. of L

<u>Micromorphism</u> (A. Weinstein):

$L_1, X_1 \overset{f}{\dashrightarrow} L_2, X_2 \quad = \text{Lagr. submfld } L_f \subset (\bar{X}_1, -\omega_1) \times (X_2, \omega_2)$

s.t. $L_f \cap (L_1 \times L_2)$ is graph of $\tilde{L}_f : L_1 \to L_2$

<u>Classif. result for smooth CY</u>:

<u>commutative analogue</u> := $\quad M$ supermfld, $\gamma_1 \in TM_{odd}$, $[\gamma_1, \gamma_1] = 0$

<u>weak CY</u>: $\gamma_2, \gamma_3, \dots \qquad \gamma_k \in \Gamma(\Lambda^k TM), \quad [\sum_{i \geq 1} \gamma_i, \sum_{i \geq 1} \gamma_i] = 0$

$\updownarrow$

<u>more classical</u>: $\omega_2, \omega_3, \dots, \ \omega_k \in \Gamma(\Lambda^k T^*M), \quad (d + L_{\gamma_1})(\sum \omega_i) = 0$

These 2 MC eqns are related by Legendre transform assuming γ_2 nondeg.
(this is the secret motivation for weak CY)