M. Abouzaid - III

$\mathcal{A}$ A_∞-algebra ; $H^*(\mathcal{A})$ graded assoc. alg. (view as A_∞, with $m_d = 0$ for $d \neq 2$)

Def: $\mathcal{A}$ is <u>formal</u> if $\exists$ A_∞-equiv^{ce} $\mathcal{A} \xrightarrow{\sim} H(\mathcal{A})$

<u>Criterion for formality</u>:

<u>Observe</u>: A graded assoc. alg., $b_\chi : A \to A$ <u>Euler vector field</u>
$$b_\chi(x) = \deg(x) \cdot x$$
Since $\deg(xy) = \deg x + \deg y$, b_χ defines a class in $HH^1(A,A)$
(since $b_\chi(x \cdot y) - x \cdot b_\chi(y) - b_\chi(x) \cdot y = 0$)

<u>Assume</u> $\exists$ closed elt $b \in CC^1(\mathcal{A},\mathcal{A}) = \mathrm{Hom}(\bigoplus_{i \geq 0} \mathcal{A}^{\otimes i}[i], \mathcal{A})$ st.
$\ni b = (b_0, b_1, b_2, \dots)$
1) $b^0 = 0 \in \mathcal{A}$
2) $b^1 \in \mathrm{Hom}(\mathcal{A},\mathcal{A})$ induces the Euler vector field on $H^*(\mathcal{A})$

Then $\mathcal{A}$ is formal (over field of char. 0)
(The converse is true, taking image of b_χ under $H^*(\mathcal{A}) \xrightarrow{\sim} \mathcal{A}$ if formal)

<u>Moreover</u> $\exists$ bijection between classes of such b and quasi-isoms. $\mathcal{A} \to H^*(\mathcal{A})$ up to equivalence.

<u>Idea of proof.</u>

Consider action of K^* on $\mathcal{A}$, by t^i on $\mathcal{A}^i$
Conjugate the A_∞-operations m_d by this action. (so: m_d mult^d by t^{2-d})
Observe: if $m_d = 0$ for $d \neq 2$ then the A_∞-str. is fixed by this action.
<u>So</u>: use higher terms of b to find a formal diffeo of $\mathcal{A}$ and obtain that the A_∞-structure is K^*-invariant.

The slogan is: Purity $\Rightarrow$ Formality
$\hookrightarrow$ ie., class in HH^1 induces action on $H^*(\mathcal{A})$ which $\equiv$ Euler v.f.

(In other terms: $H^i(\mathcal{A}) = \bigoplus_\lambda H^i_\lambda(\mathcal{A})$
by eigenvalues of b-action; purity = only have H^i_i).

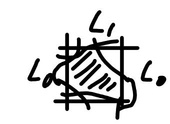
Examples: 1) Deligne-Griffiths-Morgan-Sullivan

If Q is a compact Kähler mfld then $C^*(Q;\mathbb{Q})$ is formal

for our purposes, note $C^*(Q;\mathbb{Q}) \underset{\uparrow}{\simeq} CF^*(Q,Q)$ for 0-section in T^*Q
A_∞-equivalence

2) Also note S^1 is formal.

Lekili-Perutz: $M = T^2 - \{pt\}$  $\mathcal{A} \subset \mathcal{F}(M)$ subcat. with objs L_0, L_1.

$H^*(\mathcal{A})$ = quiver with relations $L_0 \bullet \overset{u}{\underset{v}{\rightleftarrows}} \bullet L_1 \;/\; \begin{matrix} uvu = 0 \\ vuv = 0 \end{matrix}$ 

A_∞-operations: all discs w/ ∂ on L_0, L_1 are constant – no polygons

Naive hope: perturb them away (except for m_2) & show $\mathcal{A}$ is formal.

Thm (Lekili-Perutz): $\mathcal{A}$ is not formal!
(over a field of char. 0, cannot eliminate m_6 & m_8)

(Rmk: Seidel had already shown non-formality for hyperelliptic chain
on surface of genus ≥ 3 with 2 punctures)

So: absence of holom. discs doesn't guarantee formality!

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* Nontrivial example of a formal algebra appearing in Fukaya cats. [A.-Smith]
(motivation: symplectic Kh = Kh)

$\mathcal{Y}_n = \mathrm{Hilb}^0_n(A_{2n-1})$ where $A_{2n-1} = \{p(z) = x^2+y^2\} \subset \mathbb{C}^3$
$z \downarrow$ $\mathbb{C}$ — $\deg p = 2n$, with simple roots.

Hilb^0_n = consider only those subschemes whose proj. to $\mathbb{C}$ also has length n
(this is an affine open subset of Hilb_n)

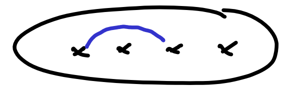
Eg. $\mathcal{Y}_1 = \mathrm{Hilb}_1(A_1) = T^*S^2$

NB: $\mathcal{Y}_n$ also has an interpretation as nilpotent slice ... rep theory of $\mathfrak{sl}_{2n}$.

Goal: find an interesting collection of Lagrangians in $\mathcal{Y}_n$.

- Lagrangians in A_{2n-1} = matching spheres (Donaldson, Seidel)

  Assume roots of p are real, then



  $\forall$ path γ connecting roots of p,
  obtain Lagr. sphere $L_\gamma \subset A_{2n-1}$

- $$\begin{tikzcd} (A_{2n-1})^n \arrow[dr] & & \mathrm{Hilb}^n_0(A_{2n-1}) = \mathcal{Y}_n \arrow[dl] \\ & \mathrm{Sym}^n(A_{2n-1}) & \end{tikzcd}$$

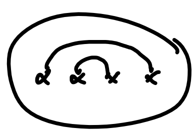
  Hilbert-Chow; bijective away from diagonal.

Given n non intersecting paths in the upper half plane connecting the zeros of p, i.e. a crossingless matching C, get $L_C \subset \mathcal{Y}_n$

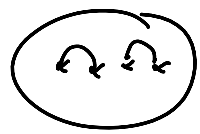
"$L_C = \prod_{\gamma \in C} L_\gamma$".

$L_C \simeq (S^2)^n$.

Eg: $n=2$:



and



In general, # of these = Catalan #.

$\mathcal{A}$:= subcat of Lagrangians in $\mathcal{Y}_n$ obtained from crossingless matchings.

Obs. (Seidel-Smith) || At the level of cohomology this looks like Khovanov's arc algebra.

Khovanov: The arc algebra has "many" nontrivial A_∞-deformations which preserve multiplication but introduce higher order terms

(rank $HH^2 \to \infty$ as $n \to \infty$)

So formality is far from automatic!

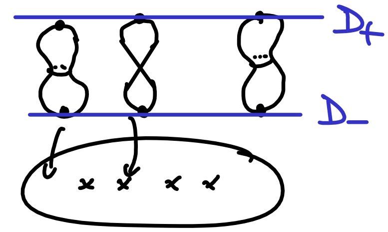
Thm (A.-Smith) || $\mathcal{A}$ is formal over $\mathbb{Q}$.

Main idea: construct a class $b \in SH^1(\mathcal{Y}_n)$ $\left(\xrightarrow[\simeq]{CO} HH^1(\mathcal{A},\mathcal{A})\right)$

& show formality criterion holds

b comes from a "propification" of projection map $\mathcal{Y}_n \to \mathrm{Conf}_n(\mathbb{C})$ induced by $A_{2n-1} \xrightarrow{z} \mathbb{C}$.

Start by propification of
$$\begin{array}{c} A_{2n-1} \subset \overline{A_{2n-1}} \\ \quad\quad\quad \downarrow \\ \quad\quad\quad \mathbb{C} \end{array}$$



$\rightsquigarrow \mathcal{Y}_n \subset \mathrm{Hilb}^0_n(\overline{A_{2n-1}})$

- If M is compact then $SH^*(M) = H^*(M)$ (no orbits at ∞!)

There is something similar if $M = \overline{M} \setminus D$, namely

Expect to have a spectral sequence

$$H^*(M) \oplus \hbar \cdot H^*(D)[\underset{\text{even}}{\hbar}, \underset{\text{odd}}{\theta}] \Rightarrow SH^*(M)$$

$\hbar$ corresponds to order of tangency to D

θ corresponds to S^1 of meridian loops around D.

Look at $\hbar$ part of this (ie. orbits going once around base  and find the class b as coming from $1 \in \hbar H^*(D)$.