Quadratic differentials of exponential type and stability

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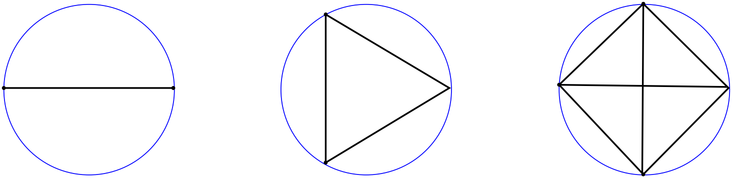
Simplest Example

Q — orientation of A_n Dynkin diagram, e.g.

$$\underbrace{\bullet \longleftarrow \bullet \longleftarrow \cdots \longleftarrow \bullet}_{n \text{ vertices}}$$

$D^b(A_n)$ — bounded derived category of reps of Q, has $\binom{n+1}{2}$ indecomposable objects up to shift

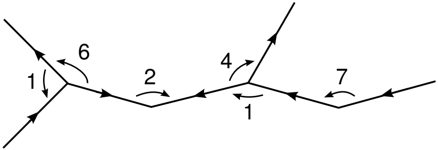


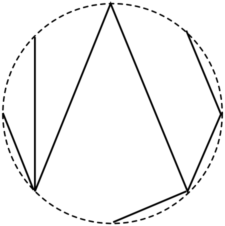
Classification of t-Structures

$\mathcal{A} \subset D^b(A_n)$ heart of a bounded t-structure, then

- $\mathcal{A}$ is artinian
- $\mathcal{A}$ has n simple objects

These form a tree, embedded in the disc. We also need to record degrees of morphisms between them.





Classification of Stability Conditions

For $\mathcal{C} = D^b(A_n)$:

stability condition $=$ *t-structure* $+$ n *numbers in* $\mathbb{H}$

One can use results of a 1932 paper of R. Nevanlinna to prove:

$$\begin{aligned}\mathrm{Stab}(\mathcal{C})/\mathrm{Aut}(\mathcal{C}) &\cong \left\{e^{P(z)}dz^2, \deg P = n+1\right\} \Big/ \mathrm{Aut}(\mathbb{C}) \\ &\cong \left\{e^{z^{n+1}+a_{n-1}z^{n-1}+\ldots+a_0}dz^2\right\} \Big/ \mathbb{Z}/(n+1)\end{aligned}$$

(as sets, also as stacks for $n > 1$)

Bridgeland Stability Conditions

Given a triangulated category $\mathcal{T}$, homomorphism $\mathrm{cl} : K_0(\mathcal{T}) \to \Gamma$, Γ finitely generated, a stability condition is

- $Z : \Gamma \to \mathbb{C}$, the *central charge*
- $\mathcal{A} \subset \mathcal{T}$, the heart of a bounded t-structure

satisfying

- Z is a stability function on $\mathcal{A}$
- Harder-Narasimhan property
- support property

Bridgeland Stability Conditions

Stability function: $Z(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$ for $0 \neq E \in \mathcal{A}$
so $\phi(E) = \mathrm{Arg}(Z(E)) \in (0, \pi]$

$0 \neq E \subset \mathcal{A}$ is semistable (resp. stable) if

$$0 \subsetneq F \subsetneq E \implies \phi(F) \leq \phi(E) \quad \text{(resp. } \phi(F) < \phi(E))$$

H.-N. property:

$$0 \neq E \subset \mathcal{A} \implies \exists \quad 0 = E_0 \subset E_1 \subset \ldots \subset E_n = E$$

with $F_i = E_i/E_{i-1}$ semistable, $\phi(F_{i-1}) > \phi(F_i)$, $0 < i \leq n$.

Support property:

$$\exists C > 0 : \quad E \text{ semistable} \implies \|\mathrm{cl}(E)\| \leq C|Z(E)|$$

Space of Stability Conditions

$\mathrm{Stab}(\mathcal{T})$ — set of stability conditions on $\mathcal{T}$

Facts:

- has structure of a complex manifold of dimension $\mathrm{rk}(\Gamma)$
- $\mathrm{Aut}(\mathcal{T})$ acts holomorphically
- $\widetilde{\mathrm{GL}^+(2,\mathbb{R})}$ acts smoothly

Dimension One

Consider:

$\mathcal{T}$ — $\mathbb{Z}$-graded Fukaya-type category of a Riemann surface

Expectation: $\mathrm{Stab}(\mathcal{T})/\mathrm{Aut}(\mathcal{T})$ is related to a space of quadratic differentials φ with prescribed critical points, equivalently flat surfaces with prescribed singularities, up to equivalence, such that

$$\text{central charge} \longleftrightarrow \text{contour integrals over } \sqrt{\varphi}$$
$$\text{stable objects} \longleftrightarrow \text{finite length geodesics of } |\varphi|$$

Quadratic Differentials of Exponential Type

Fix

- S — smooth surface (compact, connected)
- $M \subset S$ — set of marked points (non-empty, finite)
- Positive integer $n(p)$ for each $p \in M$
- Smooth, non-vanishing section of $(T^*M, J)^{\otimes 2}$ over $S \setminus M$ for some complex structure J, up to homotopy

Consider pairs (C, φ)

- C — complex curve with underlying surface S
- φ — Holomorphic section of $(T^*C)^{\otimes 2}$ over $C \setminus M$, nonvanishing, such that near $p \in M$:

$$\varphi = e^{z^{-n(p)}} h(z) dz^2$$

in some coordinate z, h meromorphic.

Geometric Origin

Given

- C — complex curve, $M \subset C$ marked points
- f — meromorphic function on C, holomorphic away from M ("LG potential")
- η — meromorphic quadratic differential on C, without zeros/poles on $C \setminus M$ ("CY structure")

then

$$e^f \eta$$

is a quadratic differential of exponential type.

However, not all are obtained this way, but always locally of this form by definition.

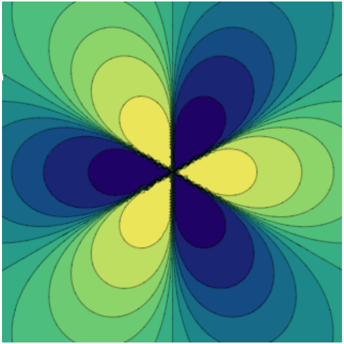
Flat Geometry

(C, M, φ) exponential type $\longrightarrow$ flat surface S_{sm}:

- underlying surface $C \setminus M$
- metric tensor $|\varphi|$

Incomplete as metric space!

Completion $S = S_{\text{sm}} \cup S_{\text{sg}}$ has a_i new points replacing p_i.

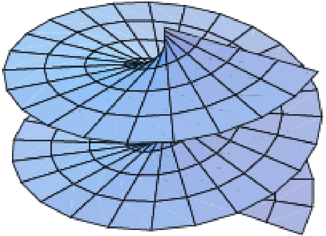


Infinite Angle Singularity

Extra points in completion are *infinite-angle singularities* of S.

Local Model: universal cover of $\mathbb{R}^2 \setminus \{0\}$ with additional point S over the origin

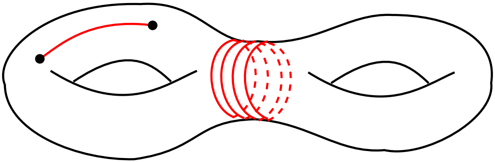
Note: $\mathbb{R}$-torsor of geodesics starting at S, geodesics can meet at S in any angle $\in \mathbb{R}$



Finite Geodesics

Two types of (maximal) geodesics of finite length on flat surface:

1. Saddle connections
   - endpoints in S_{sg}
   - rigid
2. Closed loops
   - 1-parameter families foliating cylinder



Horizontal Foliation

Flat metric $\longrightarrow$ horizontal foliation
In terms of quadratic differential φ:

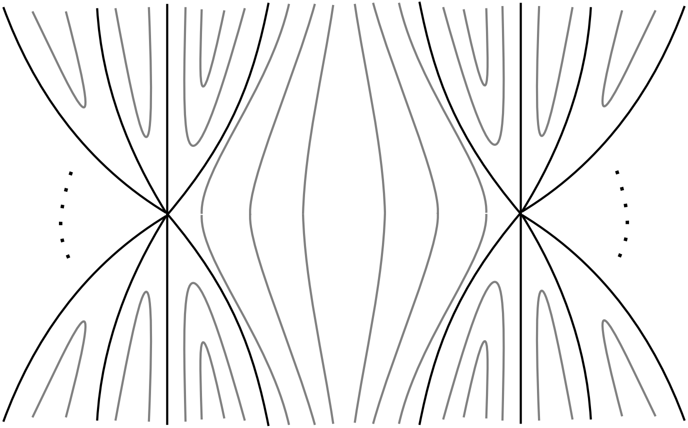
$$\varphi(v, v) \in \mathbb{R}_{\geq 0}$$

critical leaves: converge towards $S_{\rm sg}$

(C, M, φ) of exponential type, no finite leaves (generic), then critical leaves cut S into rectangular pieces:

- $\mathbb{R} \times (0, h)$ — finite number
- $\mathbb{R} \times (0, \infty)$ — infinite number

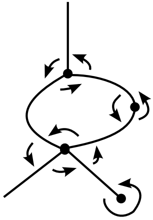
Horizontal Foliation



Ribbon Graphs

Formal definition: triple (H, σ, ι)

- H — set (maybe infinite)
- σ — bijection on H
- ι — involution on H

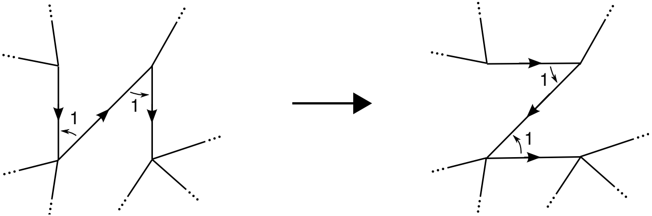


Terminology:

H	half-edges
$E = H/\iota$	edges
$V = H/\sigma$	vertices
H^ι	open edges
$E \setminus H^\iota$	proper edges

Mutation

There is a notion of left/right mutation along an edge of a ribbon graph.



Formal definition: Given a proper edge $\{h_1, h_2\} \subset H$, i.e. $\iota(h_1) = h_2$, let T be the involution

$$T = (h_1, \sigma(h_2))(h_2, \sigma(h_1))$$

then the left-mutated graph is $(H, T^{-1}\sigma T, \iota)$.

Mutation

For trivalent ribbon graphs, this is essentially quiver mutation (for special quivers), where

Graph	Quiver
proper edges	vertices
$\sigma(h_1) = h_2$	arrow from $\bar{h}_1$ to $\bar{h}_2$

Common generalization?
On categorical level:

$$F \longrightarrow L_E F \longrightarrow \mathrm{Hom}^1(E, F) \otimes E \longrightarrow F[1]$$

c.f. Kontsevich–Soibelman categorification of cluster mutation

Ribbon Graphs of Exponential Type

Dual to $\Gamma = (H, \sigma, \iota)$ is $\Gamma^* = (H, \sigma \circ \iota, \iota)$

Valency of vertex $v \in H/\sigma$ = cardinality of v as σ-orbit, so

$$\mathrm{val}(v) \in \{1, 2, \ldots, \infty\}$$

$\Gamma = (H, \sigma, \iota)$ is of exponential type if

1. finite set of vertices
2. finite set of proper edges
3. all vertices have valency$=\infty$
4. all vertices of dual graph have valency$=\infty$

Correspondence Between Flat Surface and Ribbon Graph

Consider ribbon graphs with “central charge”

$$Z(E) \in \mathbb{H}, \qquad E \text{ a proper edge of } \Gamma$$

Then:

Surface	Graph
singular points	vertices
pieces $\mathbb{R} \times (0, h)$	proper edges
vector between singular points	$Z(E)$
pieces $\mathbb{R} \times (0, \infty)$	open edges
gluing along critical leaves	σ

Γ can be embedded in S as a deformation retract (follow leaves).

Reconstruction of (C, M, φ) — Meromorphic Case

Before dealing with exponential case, consider simpler correspondence:

finite ribbon graph $\Gamma \longleftrightarrow$ meromorphic differential φ

Graph Γ	Differential φ
vertex of val. 1	simple pole
vertex of val. 2	regular point
vertex of val. $k \geq 3$	zero of order $k-2$
vertex of Γ^*	pole of order ≥ 2

order of pole for vertex v of Γ^* is:

$$2 + \# \text{ of open edges attached to } v$$

Reconstruction of (C, M, φ) — Exponential Case

Idea: ribbon graph of exp. type Γ is limit of finite ribbon graphs with increasing valency

$\longrightarrow$ construct quadratic differential of exp. type as corresponding limit of meromorphic quadratic differentials

This is a "glued" version of Euler's approximation

$$\exp(z) = \lim_{n \to \infty} \left(1 + \frac{z}{n}\right)^n$$

Graded Linear Category for Ribbon Graph

$\Gamma = (H, \sigma, \iota)$ — ribbon graph of exp. type

Define graded linear category $\overline{\mathcal{C}}_\Gamma$ over $\mathbb{C}$:

- Objects: set of proper edges $(H \setminus H^\iota)/\iota$
- Morphisms: basis of $\mathrm{Hom}^k(E_1, E_2)$ given by

$$\{(h_1, h_2) \,|\, h_i \in E_i, \sigma^k(h_1) = h_2\}$$

- Composition:

$$(h_2, h_3) \circ (h_1, h_2) = (h_1, h_3), \qquad (\iota(h_2), h_3) \circ (h_1, h_2) = 0$$

$\mathcal{C}_\Gamma$ = augmentation of $\overline{\mathcal{C}}_\Gamma$ (add identities)

Classification of Indecomposable Objects

Consider (one-sided) twisted complexes over $\mathcal{C}_\Gamma$
$\longrightarrow$ dg-category $\mathrm{Tw}(\mathcal{C}_\Gamma)$, its homotopy category, $\mathcal{T}$, is triangulated

Intuition from topology: indecomposable objects of $\mathcal{T}$ should correspond to certain paths on Γ.
Consequence: $\mathcal{T}$ has tame representation type, i.e. indecomposable objects form at most 1-dimensional families.

Fukaya category proof?

We take a more algebraic approach: matrix problems.

Bondarenko's Matrix Problem

Input: X — linearly ordered set with involution ι

$\longrightarrow$ additive category $B(X, \iota)$ with

- Objects: sequence of vector spaces V_x, $x \in X$, with

$$V_x = V_{\iota(x)}, \qquad \dim \bigoplus V_x < \infty$$

  and "block matrix" $B \in \mathrm{End}\left(\bigoplus V_x\right)$ with $B^2 = 0$.
- Morphism from $((V_x)_{x \in X}, B)$ to $((W_x)_{x \in X}, C)$ is element

$$T \in \mathrm{Hom}\left(\bigoplus V_x, \bigoplus W_x\right), \quad T_y^x \in \mathrm{Hom}(V_x, W_y)$$

  such that
  1. $TB = CT$
  2. $x > y$ implies $T_y^x = 0$ (T is lower triangular)
  3. $T_x^x = T_{\iota(x)}^{\iota(x)}$

Solution to Classification Problem

Bondarenko classifies objects in $B(X, \iota)$ (in terms of $k[x, x^{-1}]$-modules) in a 1975 paper, based on methods of Nazarova-Roiter.

We can reduce classification of objects in $\mathrm{Tw}(\mathcal{C}_\Gamma)$ to that of $B(X, \iota)$, where $X = (H \setminus H^\iota) \times \mathbb{Z}$.
Answer in terms of

- Strings: walks on the graph Γ, without U-turns
- Bands: closed walks on Γ, not powers, satisfying grading condition ($\leftrightarrow$ vanishing Maslov class)

Then

$$\text{indecomposables} = \text{strings} \sqcup (\text{bands} \times \text{Jordan blocks})$$

Phases

Next step: Classification of stability conditions on $\mathrm{Tw}(\mathcal{C}_\Gamma)$

Recall: Each stable object E has a phase

$$\frac{Z(E)}{|Z(E)|} \in S^1$$

Fact: Given a stability condition on any category,

phases of stable objects not dense in S^1

$\implies$ heart of t-structure is artinian (after tilting)

Closedness of Phases

For stability conditions from quadratic differentials of exp. type, the set of phases of all stable objects is closed in S^1

In geometric terms: Slopes of finite geodesics form closed set

Need to show: For arbitrary stability condition on $\mathrm{Tw}(\mathcal{C}_\Gamma)$ phases are closed, or at least not dense. Then we have

- Artinian heart
- Finite number N of simple objects

$$N = \mathrm{rk}(K_0(\mathrm{Tw}(\mathcal{C}_\Gamma))) = \# \text{ of proper edges of } \Gamma$$

Remaining Steps

$\mathrm{Tw}(\mathcal{C}_\Gamma)$ should depend only on topological data

- Genus of S
- Sequence of positive integers $n(p)$
- Maslov map, up to homotopy (element of $H^1(S \setminus M, \mathbb{Z})$-torsor)

The most basic version of the result would identify stability conditions with quadratic differentials, both up to equivalence.

Would also like to understand autoequivalences of the category, and their relation to the mapping class group of the surface with marked points.

Related Work

Gaiotto-Moore-Neitzke (2009) studied wall-crossing for meromorphic quadratic differentials with

- Simple zeros
- Poles of order ≥ 2, at least one

These correspond to trivalent ribbon graphs, which are in turn dual to ideal triangulations.

Bridgeland-Smith announced results relating this to CY-3 categories and stability conditions.

Possible Future Work

Meromorphic differentials:

- Infinite area case (at least one pole of order > 1): Expect stability conditions correspond, up to tilting, to ribbon graphs.
- Finite area case (no poles of order > 1): Qualitatively different, no longer expect heart of t-structure to be artinian. Simplest example: elliptic curve with constant differential

Higher dimensions?

- Methods developed here no longer apply, new ideas are needed.

The End

Thank you for your attention!