Y. Soibelman – 20/1/10 – Cohomological Hall algebra (w/ Kontsevich)

- Q finite quiver, $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$

<u>Ex</u>: $Q_0 = \bullet$ $\quad W = 0$ $\qquad Q_1 = \bullet \circlearrowleft x$ $\quad W \in \mathbb{C}[x]$

given (Q, W), I = set of vertices : $\Gamma := K_0(\mathrm{Rep}^{fd}(Q)) \cong \mathbb{Z}^I$.

$$\Gamma \supset \Gamma_+ = \mathbb{Z}^I_{\geq 0}$$

$\gamma = (\gamma^i)_{i \in I} \in \Gamma_+ \rightsquigarrow M_\gamma$ = space of rep^ns of dim $= \gamma$.

$$M_\gamma \circlearrowleft G_\gamma = \prod_{i \in I} GL(\gamma^i)$$

$W_\gamma = \mathrm{Tr}(W) : M_\gamma \to \mathbb{C}$, G_γ-invariant

This appears in various settings.

<u>Ex</u>: LG-models

<u>Ex</u>: matrix integrals : $\int_{C \leftarrow \text{real contour}} e^f \alpha$, $\int_C e^{f/\hbar} \alpha$ $(\hbar \to 0)$.

- Categorically, $(\mathcal{C}, W) \rightsquigarrow \mathcal{C} = \mathcal{C}_{(Q,W)}$ 3-CY category (ie. $\mathrm{Ext}^i \otimes \mathrm{Ext}^{3-i} \to \mathbb{C}$)

$\mathcal{C}$ has a t-structure with heart $= \mathbb{C}Q/\langle \partial W \rangle$-mod.
(roughly: triangulated envelope of this)

(e.g. if $W = 0$ then take path algebra, adding backward arrows to ensure 3CY property. Or: take Koszul dual, ie. arrows of Q live in degree 1; set their products $\equiv 0$; add backwards arrows in deg. 2, and a deg. 3 endom. at each vertex; finally, W determines an A_∞-deformation.)

- Geometric analogy: $W = CS_{\mathbb{C}}(A) = \int_{X \atop \text{3d CY}} \mathrm{Tr}\left(\frac{\bar\partial A \wedge A}{2} + \frac{A^3}{3} \right) \wedge \Omega^{3,0}$
(holom. Chern-Simons) $\qquad$ (A (0,1)-connection)

Then $\mathrm{crit}(W)$ = holomorphic structures (+ stability)

- for $\gamma \in \Gamma$, $\Omega(\gamma) :=$ "# semistable objects in class γ"

  $\Omega^{mot}(\gamma) :=$ "Serre polynomial of vanishing cycles of W at $\mathcal{C}^{ss}_\gamma$".

  cf. talks last year on motivic DT-invariants

<u>Today:</u> <u>alternative approach</u> to DT and motivic DT-inv'ts
(cf. 0811.2435)

New object: $\mathcal{H} = \mathcal{H}^{(Q,W)} = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_\gamma$ "cohomological Hall algebra"

I) <u>For $W = 0$:</u>

$\mathcal{H}_\gamma = H^\bullet_{G_\gamma}(M_\gamma)$ equiv't cohom. of rep's of quiver Q.

$m_{\gamma_1,\gamma_2} : \mathcal{H}_{\gamma_1} \otimes \mathcal{H}_{\gamma_2} \to \mathcal{H}_{\gamma_1+\gamma_2}$ <u>multiplication map</u>

Nakajima-type def^n, as follows:

- Let $M_{\gamma_1,\gamma_2} \subset M_\gamma$ for $\gamma = \gamma_1 + \gamma_2$ (i.e. $\gamma^i = \gamma_1^i + \gamma_2^i$)

  $:=$ representations of Q of dim. γ s.t. $(\mathbb{C}^{\gamma_1^i}) \subset \mathbb{C}^{\gamma^i}$ form a subrepresentation of dim. γ_1.

The sub-gauge group $G_{\gamma_1,\gamma_2} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$ acts on M_{γ_1,γ_2}

$$H^\bullet_{G_{\gamma_1} \times G_{\gamma_2}}(M_{\gamma_1} \times M_{\gamma_2}) \xrightarrow{\sim} H^\bullet_{G_{\gamma_1,\gamma_2}}(M_{\gamma_1,\gamma_2}) \longrightarrow$$

$$\longrightarrow H^{\bullet + \text{shift}_1}_{G_{\gamma_1,\gamma_2}}(M_{\gamma_1+\gamma_2}) \longrightarrow H^{\bullet + \text{shift}_1 + \text{shift}_2}_{G_{\gamma_1+\gamma_2}}(M_{\gamma_1+\gamma_2})$$

gives product; total shift $= \chi_Q(\gamma_1,\gamma_2) = \dim \mathrm{Hom} - \dim \mathrm{Ext}^1$
Euler form

<u>Thm:</u> (m_{γ_1,γ_2}) defines an <u>associative product</u>

<u>Ex.:</u> $Q_d =$  d loops $\Rightarrow \gamma = n \in \Gamma = \mathbb{Z}$

$\mathcal{H} = \bigoplus_{n \geq 0} H^\bullet(BGL(n))$ indep't of d since no relations in quiver
$\Rightarrow G_\gamma = GL(n)$ action on $\{d\text{-tuples of } \mathrm{Mat}_{n \times n}\}$
$=$ contractible

but product depends on d.

(Ex. cont'd)

Explicit formula: (recall $H^\bullet(BGL(n))$ = symmetric polynomials in n variables)

$$(f_1\cdot f_2)(x_1,\dots,x_{n+m}) = \sum_{\substack{i_1<\dots<i_n \\ j_1<\dots<j_m \\ \{i_1\dots i_n, j_1\dots j_m\}=\{1\dots n+m\}}} f_1(x_{i_1}\dots x_{i_n})\, f_2(x_{j_1}\dots x_{j_m}) \cdot \prod_{k=1}^{n}\prod_{l=1}^{m} (x_{j_l}-x_{i_k})^{d-1}.$$

- $d=0$: $\Rightarrow$ $\mathcal{H}$ = free fermion $= \Lambda[\psi_1,\psi_3,\cdots]$
- $d=1$: $\mathcal{H}$ = free boson algebra = gen^d $\varphi_0, \varphi_2, \varphi_4, \cdots$

Now can write the generating polynomial

$$P_{(Q_d,W=0)}(z,q^{1/2}) = \sum_{n,m} \dim \mathcal{H}_{n,m}\, z^n q^{m/2} = \sum_{n\geq 0} \frac{q^{\frac{(1-d)n^2}{2}} z^n}{(1-q)\cdots(1-q^n)}$$

"quantum dilogarithm"

<u>Prop:</u> $P_{(Q_d,W=0)}(z,q^{1/2}) = A^{mot}(z,q^{-1/2})$

$\uparrow$ motivic DT-inv^t of the 3-CY cat. $\mathcal{C}_{(Q_d,W=0)}$

(general phenomenon, not just for this example).

II) <u>When $W\neq 0$</u>:

Recall we have (M_γ, W_γ) — M_γ: rep^n variety; W_γ: trace of W in given rep^n. (G_γ-inv^t)

Recall: X alg var$/\mathbb{C}$, $f\in\mathcal{O}(X)$ $\rightsquigarrow H^\bullet(X,f)$. Two realizations:

- $H^\bullet_B(X,f) := H^\bullet(X, f^{-1}(t))$, $t \ll 0$.
  "rapid decreasing cohomology"
- $H^\bullet_{DR}(X,f) = \mathbb{H}^\bullet(X,(\Omega^\bullet_X, d+df\wedge\cdot))$
  ie. look at diff^l forms st. $d(e^f\eta)=0$, with decay condition at $\mathrm{Re}\, f\to -\infty$

Def. $\mathcal{H}_{(Q,W)} := \bigoplus_{\gamma \in \Gamma} H^{\bullet}_{G_\gamma}(M_\gamma, W_\gamma)$

e.g. for $\bullet\circlearrowleft$, Ⓝ, get $N+1$ tensor power of infinite wedge.

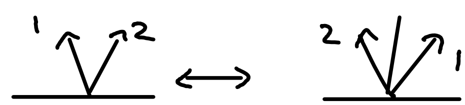
- $\mathcal{H}^{(Q,W)}$ depends on t-structure, but not on stability cond.
- $Z : \Gamma \to \mathbb{C}$ central charge
  $\Rightarrow M_\gamma$ can be filtered by $M_{\gamma; \gamma_1 \dots \gamma_n} = \{$ reps of fixed Harder-Narasimhan type $\}$
  $\Rightarrow$ spectral sequence converging to $\mathcal{H}_\gamma$.

$$A = A^{(Q,W)} := \sum_{\gamma \in \mathbb{Z}^I_{\geq 0}} [\mathcal{H}_\gamma]\, e_\gamma \in \text{quantum torus}$$
$$e_{\gamma_1} e_{\gamma_2} = q^{\chi_Q(\gamma_1,\gamma_2)} e_{\gamma_1+\gamma_2}.$$

Prop: $A = \overrightarrow{\prod}_{\gamma_i \text{ primitive vector}} A^{ss}_{\gamma_i}$

where $A^{ss}_{\gamma_i} = \sum_{n \geq 0} \left[H^{\bullet}_{G_{n\gamma_i}}(M^{ss}_{n\gamma_i}) \right] e^n_{\gamma_i}$ (semistable reps.)

Eg: A_2-case:



$$\mathcal{H} \simeq \mathcal{H}_1 \otimes \mathcal{H}_2 = \mathcal{H}_2 \otimes \widetilde{\mathcal{H}} \otimes \mathcal{H}_1.$$

- let $(x;q)_\infty := (1-x)(1-qx)(1-q^2x)\dots$

Def: $F \in \mathbb{Q}((q^{1/2}))[[x]]$ is <u>admissible</u> if can be factorized into
$$F = \prod_{n \geq 1} \prod_{i \in \mathbb{Z}} (q^{i/2} x^n; q)_\infty^{c(n,i)} \quad \text{for some } c(n,i) \in \mathbb{Z},$$
and where for given n all $c(n,i) = 0$ for $|i| \gg 0$.

Can generalize to "quantum admissible" when $x_i x_j = q x_j x_i$

Thm: Applying Serre polynomial to A we obtain a quantum admissible series.