V. Ginzburg – 18/1/10 – Gerstenhaber–BV structures on coisotropic intersections

(w/ Baranovsky)

X smooth alg./$\mathbb{C}$ $\supset Y, Z$ smooth subvarieties

$\mathrm{Tor}(\mathcal{O}_Y, \mathcal{O}_Z)$ graded comm. algebra $\quad (\mathrm{Tor} = \mathrm{Tor}^{\mathcal{O}_X})$.

$\mathrm{Ext}(\mathcal{O}_Y, \mathcal{O}_Z)$ graded module

<u>Assume:</u> X alg. Poisson structure $\{-,-\}$ given by $P \in H^0(\Lambda^2 T_X)$

Y, Z coisotropic subvars., i.e. $\{\mathcal{I}_Y, \mathcal{I}_Y\} \subseteq \mathcal{I}_Y$.

($\Leftrightarrow$ the image of P in $\Lambda^2 N_{X/Y}$ is zero).

<u>Thm 1:</u>
1) For any coisotropic $Y, Z \subset X$, $\mathrm{Tor}_*(\mathcal{O}_Y, \mathcal{O}_Z)$ has a canonical Gerstenhaber algebra structure
2) $\mathrm{Ext}^\bullet(\mathcal{O}_Y, \mathcal{O}_Z)$ has a canonical structure of Gerstenhaber module over $\mathrm{Tor}(\mathcal{O}_Y, \mathcal{O}_Z)$

- Gerstenhaber algebras are often BV-algebras: ie.

$\exists\, \Delta : A^\bullet \to A^{\bullet - 1}$ diff^l operator of order ≤ 2, with $\Delta^2 = 0$,

and s.t. $\{a, b\} = \Delta(ab) - \Delta(a) b - (-1)^{|a|} a \Delta(b)$

<u>Ex:</u> (Y, P) Poisson manifold. $X = Y \times Y \supset Y = Z =$ diagonal.

$\mathrm{Tor}_\bullet(\mathcal{O}_Y, \mathcal{O}_Y) \cong \Lambda^\bullet T_Y^* = \Omega_Y^\bullet$

$\mathrm{Ext}^\bullet(\mathcal{O}_Y, \mathcal{O}_Y) \cong \Lambda^\bullet T_Y$

$\Delta : \Omega^\bullet \to \Omega^{\bullet - 1}$ BV-operator is given by

$$\Delta = \iota_P d_{dR} + d_{dR} \iota_P \qquad \begin{pmatrix} \iota_P = \text{contract w/ Poisson bivector} \\ d_{dR} = \text{de Rham diff}^l. \end{pmatrix}$$

$\rightsquigarrow$ induced Gerstenhaber bracket on $\Omega_Y^\bullet$ is the Koszul bracket.

<u>Now assume:</u> X is (holom.) symplectic, Y, Z are Lagrangian

$\mathcal{L}_Y = K_Y^{1/2}$ half-forms, similar for Z.

Then: <u>Thm 2:</u> $\exists$ canonical BV-operator $\Delta : \mathrm{Ext}^\bullet(\mathcal{L}_Y, \mathcal{L}_Z) \to \mathrm{Ext}^{\bullet - 1}(\mathcal{L}_Y, \mathcal{L}_Z)$

Motivation comes on one hand from Behrend-Fantechi; on the other hand, from a conj. of Kapustin & Rozansky.

Conj: (Kapustin-Rozansky)

given Y, Z Lagr. in alg. symplectic X, $\exists$ triangulated cat. $\mathcal{C} = \mathcal{C}(\mathcal{L}_X, \mathcal{L}_Y)$ st.
$\mathcal{L}_Y = K_Y^{1/2}$ $\mathcal{L}_Z = K_Z^{1/2}$

1) $HH_\bullet(\mathcal{C}) = Ext^\bullet(\mathcal{L}_Y, \mathcal{L}_Z)$, with Connes diff.! on $HH_\bullet \longleftrightarrow \Delta$ (cf Thm 2)

2) $HH^\bullet(\mathcal{C}) = Tor_\bullet(\mathcal{O}_Y, \mathcal{O}_Z)$, with Gerstenhaber bracket $\longleftrightarrow \{\cdot,\cdot\}$ (cf. Th. 1)

Only understood in the following special case:

- $X = T^*Y \supset Y$ = zero section
  $Z = graph(df)$, $f \in \mathcal{O}(Y)$

  $Y \cap Z$ = critical locus of f.

  Using Koszul resolution, $Tor_\bullet(\mathcal{O}_Y, \mathcal{O}_Z) \simeq \mathcal{H}^*(\Lambda^\bullet T_Y^*, \wedge df)$

  $Ext(\mathcal{O}_Y, \mathcal{O}_Z) = \mathcal{H}^*(\Lambda^\bullet T_Y, i_{df})$

  $d_{DR} \curvearrowright \Lambda^\bullet T_Y^*$ anticommutes with $\wedge df$

  $\Rightarrow d_{DR}$ induces a BV differential on $\mathcal{H}^\bullet(\Lambda^\bullet T_Y^*, \wedge df) = Tor$

- Kapustin-Rozansky define $\mathcal{C}(\mathcal{O}_Y, \mathcal{O}_{graph(df)})$ = category of matrix factorizations of f.

  (recall $MF = E^+ \underset{\partial^-}{\overset{\partial^+}{\rightleftarrows}} E^-$, $E^\pm$ vector bundles on Y,
  $\partial^+\partial^- = f\cdot id$, $\partial^-\partial^+ = f\cdot id$)

  If f has an isolated singularity,

  $\mathcal{H}^\bullet(\Lambda^\bullet T_Y, i_{df}) = Coker\left[T_Y \xrightarrow{i_{df}} \mathcal{O}_Y\right] = Jac(f)$ Jacobian ring

  & we know $HH_\bullet(\mathcal{C}) \simeq Jac(f)$ ✓ ($HH^\bullet = HH_\bullet$ here).

- Main tool in thms: deform. quantization:

$\mathcal{O}_X^\varepsilon$ noncomm. deformation of $\mathcal{O}_X$ over $\mathbb{C}[\varepsilon]/\varepsilon^2$

$$f * g = fg + \frac{\varepsilon}{2}\{f,g\}$$ deformⁿ induced by P.

$Y, Z \subset X$ coisotropic

$\mathcal{L} \to Y$ line bundle $\rightsquigarrow$ deform to $\mathcal{L}^\varepsilon$ left modules over $\mathcal{O}_X^\varepsilon$?

$\mathcal{M} \to Z$ —"— $\mathcal{M}^\varepsilon$

$\rightsquigarrow$ can look at $\mathrm{Tor}^{\mathcal{O}_\varepsilon}(\mathcal{L}^\varepsilon, \mathcal{M}^\varepsilon)$.

- Let Y = Lagrangian submfld in symplectic X, $\mathcal{L}_Y = K_Y^{1/2}$

Additional data: Lagrangian splitting of exact seq.

i.e. $0 \to N^*_{Y/X} \underset{p}{\overset{\longleftarrow}{\longrightarrow}} T^*X|_Y \to T^*_Y \to 0$ st. $(\ker p)^\perp$ Lagrangian.

conormal bundle to Y.

Then given $f \in \mathcal{O}_X$, $df \rightsquigarrow p(df)$ section of $N^*_{Y/X} \rightsquigarrow \xi_{p(df)} \in T_Y$.

$\rightsquigarrow$ get quantization of $\mathcal{L}_Y$: $f * l = f.l + \frac{\varepsilon}{2} L_{\xi_{p(df)}} l$

(Note: any two Lagrangian splittings are fiberwise homotopic).

This is the main input into Thm 2.

- Let Y = coisotropic submfld of X, $\mathcal{L} \to Y$ line bundle.

$\to P \in H^0(\Lambda^2 T_X)$ Poisson bivector induces $\overline{P} \in H^0(Y, N_Y \otimes T_Y)$

(using: Y coiso. so P vanishes in $\Lambda^2 N_Y$. ✓)

$\to \kappa \in H^1(X, T_X)$ measuring the nonsplitness of $\mathcal{O}_X \to \mathcal{O}_X^\varepsilon \to \mathcal{O}_X$

$\to \alpha(N_{X/Y}) \in H^1(Y, (\mathrm{End}\, N) \otimes \Omega^1_Y)$ Atiyah class.

<u>Prop:</u> $\mathcal{L}$ deforms to a left $\mathcal{O}_X^\varepsilon$-module iff

$$\overline{P} \cup [2\,\mathrm{id}_N \otimes c_1(\mathcal{L}) - \alpha(N)] + \overline{\kappa} = 0 \quad \text{in } H^1(Y, N)$$

$$H^0(Y, N \otimes T_Y) \otimes \mathrm{Ext}^1(N \otimes T_Y, N) \to H^1(Y, N)$$

For Y Lagrangian, this condition becomes $2c_1(\mathcal{L}) = c_1(K_X)$, hence half forms.