T. Dyckerhoff – 23/1/10 – Isolated hypersurface singularities as noncomm. spaces

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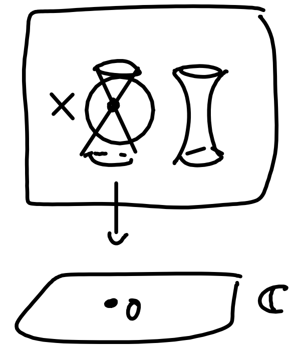
1. Context:

$w: \mathbb{C}^n \to \mathbb{C}$ polynomial with isolated sing. at 0



Milnor '69: for $t \neq 0$, fiber $X_t \underset{h.e.}{\sim} \bigvee_{\mu} S^{n-1}$

Milnor number $\mu = \dim_{\mathbb{C}} \mathbb{C}[[x_1, \dots x_n]] \Big/ \left(\frac{\partial w}{\partial x_1}, \dots, \frac{\partial w}{\partial x_n}\right)$

(Jacobian ring = Milnor ring).

Algebraic setup:
- $R = \mathbb{C}[[x_1, \dots x_n]]$ regular local ring $\supset m$ max. ideal
- $w \in m^2$ isolated singularity, ie. $\left(\frac{\partial w}{\partial x_1} \dots \frac{\partial w}{x_n}\right)$ is regular
- $S = R/w$

$\Rightarrow$ nc-geometry by studying the dg-category $D^b_{sing}(S) = D^b_{coh}(S) / Perf(S)$ (Buchweitz, Orlov: category of singularities).

In hypersurface case there's also a nice model, namely matrix factorizations.

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2. Homological algebra of hypersurfaces (Eisenbud 80's)

a) warmup: R regular local ring, M coherent R-module

$\Rightarrow \operatorname{proj\,dim}(M) = \dim(R) - \operatorname{depth}(M)$ (Auslander-Buchsbaum)

Cor.:
- every such M has a finite free resolution
- if $\dim(R) = \operatorname{depth}(M)$ then M is free.

b) singular hypersurface: $S = R/w$.

M coh. S-module with $\dim S = \operatorname{depth} M$ (maximal Cohen Macaulay)

$\Rightarrow M$ is also an R-module, annihilated by w

$\operatorname{proj\,dim}_R(M) = 1$, so $\exists$ free resolution $(X^1 \xrightarrow{\varphi} X^0) \simeq M$

Moreover,
$$\begin{tikzcd} X^1 \arrow[r, "\varphi"] \arrow[d, "w"'] & X^0 \arrow[d, "w"] \arrow[dl, dashed, "\exists\psi"'] \\ X^1 \arrow[r, "\varphi"'] & X^0 \end{tikzcd}$$

$w = 0$ on M
$\Rightarrow \exists\, \psi$ homotopy st.
$$\varphi \circ \psi = \psi \circ \varphi = w . \mathrm{id}$$
ie. matrix factorization!

This induces a $\mathbb{Z}/2$-periodic free resolution in category of S-modules:
$$\left(\cdots \to \bar{X}^1 \xrightarrow{\bar{\varphi}} \bar{X}^0 \xrightarrow{\bar{\psi}} \bar{X}^1 \xrightarrow{\bar{\varphi}} \bar{X}^0 \right) \xrightarrow{\sim} M.$$

Thm: (Eisenbud)

Every coherent S-module has an S-free resolution which eventually becomes 2-periodic.

$\Rightarrow$ define $\mathbb{Z}/2$-dg category of matrix factorizations $MF(R,w)$.

- extend it to a $\mathbb{Z}/2$-periodic, $\mathbb{Z}$-graded category: $MF(R,w) \circlearrowleft k[u,u^{-1}]$, $\deg u = 2$

Then: $\boxed{MF(R,w) \xrightarrow{\sim} D^b_{sing}(S)}$

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3. nc-geometry

X variety$/k$ $\Longrightarrow$ $D_{qcoh}(X)$ dg-derived cat. of unbounded complexes of quasicoh. sheaves.

Properties: properness, smoothness
Invariants: De Rham cohom.
Hodge cohomology
Hodge filtration

$\Bigg\}$ can be extracted in a natural way from $D_{qcoh}(X)$ (cf. Toën's talks)

$\Rightarrow$ forget X and work purely with categories.

Do this in the context of D^b_{sing} (& MF in case of hypersurfaces)

Postulate: $\mathfrak{X}$ nc-space $\Rightarrow D_{qcoh}(\mathfrak{X}) \simeq MF^{\infty}(R,w)$

infinite rank matrix fact$^{\underline{s}}$.

This is = modules over MF !!

($\Rightarrow$ karoubi-closed, no issues w/ completion)

(NB: for isolated sing, MF is already karoubi-closed ??)

Thm: (1) $MF^{\infty}(R,w)$ has a compact generator, namely the field k^{stab} (stabilized, viewed as matrix fact^{n} by resolution ...)

$\Rightarrow MF^{\infty}(R,w) \simeq D(A)$, $A = R\mathrm{End}(k^{stab})$.

$H^{\alpha}(A) = \mathbb{Z}/2$- Clifford algebra over k.

Hence $\mathfrak{X}$ is proper over $k[u,u^{-1}]$

- 2-periodic variant of Toën/Tabuada's homotopy theory of dg-cat's:

$$\Rightarrow \left(H_0(dgcat/k[u,u^{-1}]),\ - \otimes_{k[u,u^{-1}]} -,\ \underbrace{\underline{RHom}}_{\text{internal hom.}}(-,-) \right)$$

is a closed symmetric monoidal category

Thm (cont.): (2) $\underline{RHom}_{continuous}(MF^{\infty}(R,w), MF^{\infty}(R',w'))$

$$\simeq MF^{\infty}(R \otimes R', -w \otimes 1 + 1 \otimes w')$$

(map in direction $\longleftarrow$ is: given a MF over $(R\otimes R', -w+w')$ and a MF over (R,w), tensor them together, then w's cancel out and get a MF over (R',w')).

(3) $id \in \underline{R\mathrm{End}}_c(MF^{\infty}(R,w))$ can be explicitly constructed as the stabilized diagonal bimodule Δ^{stab}. and it is compact $\Rightarrow$ hence $\mathfrak{X}$ is smooth.

Thm (cont.)

(4) $A^! \simeq A[n]$ i.e. <u>Calabi-Yau</u> category

(5) $HH_*^{\mathbb{Z}/2}(MF(R,w)) \cong R/(\partial_1 w, \dots, \partial_n w)[n]^{\mathbb{Z}/2}$

$\cong HP_*(MF(R,w))$

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4. <u>Dessert</u>: (w/ D. Murfet) explanation of pairing on $\mathrm{Hom}_{MF}(X,Y)$.

$\mathcal{X}$ is a proper smooth CY

$$\Downarrow \text{Conj.}$$

conj. (Kontsevich-Soibelman) should follow from Lurie's work on top. field theories

$\exists$ trace map $tr: A \longrightarrow k$ which factors

$$\begin{tikzcd} A \arrow[r, "tr"] \arrow[d] & k \\ A \otimes^L_{A^e} A \arrow[d] & \\ (A \otimes^L_{A^e} A)_{hS^1} \arrow[uur, dotted] & \end{tikzcd}$$

homologically non-degenerate

($\Rightarrow$ cyclic symmetry)

$T := MF(R,w)$

$T(X,Y) \otimes T(Y, X[n]) \longrightarrow \mathbb{C}$ pairing:

$$(G,F) \longmapsto \frac{1}{(2\pi i)^n} \oint_{\{|\frac{\partial w}{\partial x_i}| = \varepsilon\}} \frac{\mathrm{tr}(F \circ G \circ (dQ)^m)}{\partial_1 w \dots \partial_n w}$$

where $Q = \begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix}$ MF X.

derived by Kapustin-Li using path integrals

(formula doesn't involve MF for Y, but could do it the other way around)

$$Z \simeq T(X,Y) \Rightarrow \begin{tikzcd} R\Gamma_{\{m\}} Z \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathrm{Hom}\big(\overbrace{\mathrm{Hom}(Z,R)}^{\overset{tr}{\cong} T(Y,X)}, \underbrace{R\Gamma_{\{m\}} R}_{H^n_m(R)[-n]}\big) \\ Z & \end{tikzcd}$$

ie:

$$\begin{tikzcd} R\Gamma_{\{m\}}(T(X,Y)) \arrow[r, "\sim"] \arrow[d, "\simeq"] & \mathrm{Hom}(T(Y,X), H^n_m(R)) \arrow[d, "\mathrm{Res}_* \simeq"] \\ T(X,Y) \arrow[r, dashed, "\sim"] & \mathrm{Hom}(T(Y,X), k) \end{tikzcd}$$

homological perturb^n lemma + Koszul model

Residue

the duality pairing we want

Looking at this carefully motivates the Kapustin-Li formula from homological algebra perspective.