M. Kontsevich – 22/1/10 – Refined cohomological Hall alg & holomorphic CS

3-CY category:

- $\mathcal{E}$ object $\rightsquigarrow$ $m_n : \underline{\mathrm{Hom}}(\mathcal{E},\mathcal{E})^{\otimes n} \longrightarrow \underline{\mathrm{Hom}}(\mathcal{E},\mathcal{E})$ deg. $2-n$ $\quad \forall n \geq 1$.
- for a d-CY: $\quad (\cdot,\cdot) : \underline{\mathrm{Hom}}(\mathcal{E},\mathcal{E}) \otimes \underline{\mathrm{Hom}}(\mathcal{E},\mathcal{E}) \to k$ deg $-d$.

  $(m_n(\dots),\cdot) : \mathrm{Hom}(\mathcal{E},\mathcal{E})^{\otimes n+1} \to k$ deg. $2-d-n = (3-d)-(n+1)$.

Deformations $\rightsquigarrow$ m_0 can appear, but also when $d=3$, $\underline{m_{-1} \in k}$!
The interpretation of m_{-1} lies in CS theory.
The objects of the "right" 3 CY = critical points of a function,
m_{-1} = critical value.

X alg. var. $/\mathbb{C}$, $\quad f \in \mathcal{O}(X)$

$\Rightarrow$ finite subset of bifurcation values $\mathrm{Bif}_f = \{z_1, \dots z_m\} \subset \mathbb{C}$

<u>Def:</u> $z \notin \mathrm{Bif}_f \iff f|_{f^{-1}(D_z)}$ is a trivial fibration.

(includes critical pts of f + "non-proper" pts,
e.g. inclusion $\mathbb{C}^* \hookrightarrow \mathbb{C}$ has $\mathrm{Bif} = \{0\}$).

$\rightsquigarrow$ <u>constructible sheaf</u> of $\mathbb{Z}$-mod$/\mathbb{C}$; for $n \in \mathbb{Z}$,
stalk $\mathcal{E}_z = H^n_B(X, f^{-1}(z); \mathbb{Z})$, with property that $R\Gamma(\mathcal{E}) = 0$.

☆ <u>Equivalently</u>: such a constr. sheaf with $R\Gamma = 0$

$\iff$ $V_1, \dots, V_m$ $\mathbb{Z}$-mod ; $\forall i,j$, $T_{ij} : V_i \to V_j$, T_{ii} invertible.

Namely, $V_i := H^n(f^{-1}(D_{z_i}), f^{-1}(pt); \mathbb{Z})$

T_{ij} = induced by straight path $z_i \longrightarrow z_j$

Fixing V_i and T_{ii}, the possible choices of T_{ij}, $i \neq j$
amount to choices of paths b/w z_i's, so carry an
action of the braid group B_n.

* $\mathfrak{g} = \bigoplus_{i,j} \mathrm{Hom}(V_i, V_j)$, graded $(0,0,\dots,\underset{i}{-1},\dots,\underset{j}{+1},\dots,0) \in \mathbb{Z}^{n-1}$

Stability := real-stability by straight line path $z_i \bullet\!\!-\!\!-\!\!-\!\!\bullet\, z_j$

As the z_i's move in $\mathbb{C}$, stability cond. changes and we get wall-crossing phenomena.

* the constructible sheaf $\mathcal{E}$ is in fact a <u>mixed Hodge module</u> (M. Saito, ...)

Filtered holonomic D-module (de Rham cohom. = 0):

| constr. sheaves with $R\Gamma = 0$ form a symm. monoidal cat. using $\otimes$ convolution product.

+ weight filtration.

---

* <u>Exp. motives</u> = mixed Hodge modules with $R\Gamma = 0$.

- Fiber functors: $H^n_B(X, f) = H^n_B(X(\mathbb{C}), f^{-1}(\infty); \mathbb{Z})$

$$H_{dR}(X, f) = H_{dR}\left(X_{Zar}; \left(\Omega^\bullet_X, d + \tfrac{df}{u}\right)\right)$$

(<u>NOT</u> complex topology, for otherwise the exponential f$^{\underline{n}}$ would kill df/u).

(u = number)

- Critical cohomology:

$$\bigoplus_{z_i \text{ crit vals.}} \mathbb{R}\Gamma\left(f^{-1}(z_i), \varphi_{f - z_i}(\mathbb{Z}_X)\right)$$

$$\| \text{ over } \mathbb{C}((u))$$

$$H_{dR}\left(X_{Zar}; \left(\Omega^\bullet_X((u)), d + df/u\right)\right)$$

The two are equivalent [not written up].

* Matrix integrals:

Q finite quiver w/ vertices $1 \dots n$

$W \in \mathbb{C}Q / [\mathbb{C}Q, \mathbb{C}Q]$

for $\gamma = (\gamma^i)$, $\gamma^i \geq 0$, $\quad M_\gamma :=$ rep$^{\text{ns}}$ of Q in $(\mathbb{C}^{\gamma_i})_i$
(= $\mathbb{C}$ vector space).

M_γ carries an action of $G_\gamma = \prod GL(\gamma_i, \mathbb{C})$.

$W_\gamma = \operatorname{Tr} W$ in given representation $\in \mathcal{O}(M_\gamma)^{G_\gamma}$

Consider $\displaystyle\int_{\text{rapid decay}} \exp(W_\gamma)$.

Equivariant setting:
$$\begin{array}{c} EG_\gamma \times_{G_\gamma} M_\gamma \\ \downarrow \\ BG_\gamma \end{array} \quad \text{product of } \infty\text{-dim. Grassmannians.}$$

$\rightsquigarrow \mathcal{H}_\gamma := H^\bullet_{G_\gamma}(M_\gamma, W_\gamma)$ equiv^t cohomology

weight filtration, depends on stability condition

$\rightsquigarrow$ counting of semistable rep$^{\text{ns}}$ ($\in \mathcal{H}_\gamma$) is subject to wall-crossing.

Stability: $z_i \in \mathbb{C}$, $\operatorname{Im} z_i > 0$

$\Rightarrow M_\gamma \supset M_\gamma^{(ss, z_i)}$ open.

* can also associate to (Q, W) a 3CY cat. with t-structure whose heart $\cong \bigcup_\gamma \operatorname{crit} W_\gamma$.

(every crit pt of W_γ gives a rep$^{\text{n}}$ of Jacobian algebra; these rep$^{\text{ns}}$ form the heart of a t-structure).

* On the moduli stack of objects of 3CY, we have a sheaf of vanishing cycles.

Now consider <u>more interesting 3CY categories</u>:

1) A dg-algebra of finite type + 3CY structure
   $C =$ finite-dim. dg modules

2) A proper dg-alg., 3CY
   $C =$ Perf A-mod.

<u>Example:</u> X 3 dim. CY/$\mathbb{C}$ (possibly noncompact)
$\cup$
Z closed compact subset

$C = \mathrm{Perf}_Z(X)$.

(<u>NB:</u> case 1) corresponds to (Q, W), W polynomial
2) formal power series)

P (= path algebra), $W \in P/[P,P]$, f.d reps, crit vals.

- Stability conditions: $K_0(C) \xrightarrow[\text{fixed}]{} \mathbb{Z}^n \xrightarrow[\text{stability } Z]{} \mathbb{C}$

To get a reasonable map $K_0(C) \to \mathbb{Z}^n$ that makes the notion of stab. cond^{n} manageable, use: (in case 2)

$P_1 \ldots P_n \in \mathrm{Perf}(A\text{-mod})$ fixed set of perfect modules

$\Rightarrow K_0(C) \longrightarrow \mathbb{Z}^n$

$\mathcal{E} \in$ fin dim $\longmapsto \left(\chi(R\mathrm{Hom}(P_i, \mathcal{E}))\right)_{i=1}^{n}$

C^{ss}_γ semistable objects in class γ = stack of finite type $\forall \gamma$

we need to equip the stack C^{ss}_γ with a constructible sheaf of vanishing cycles. Such sheaf does not come for free, unlike the previous examples.

2 natural choices, living in respectively
$H^0_{\text{ét}}(\text{Moduli stack of objs}, \mathbb{Z}/2)$ and $H^1_{\text{ét}}(\ldots, \mathbb{Z}/2)$

$H^0_{\acute{e}t}(\mathcal{M}, \mathbb{Z}/2) \ni$ normalized dimension mod 2

$H^1_{\acute{e}t}(\mathcal{M}, \mathbb{Z}/2) \ni$ determinant bundle mod 2.

given moduli stacks of semistable objects + constr. sheaves, can again define critical cohomology.

★ **Holom. CS theory:**

- X smooth proper CY 3-fold $/\mathbb{C}$, given $\Omega \in \Omega^{3,0}$

$\begin{array}{c}\mathcal{E}\\ \downarrow\\ X\end{array}$ trivial C^∞-bundle, $A \in \Gamma(X, \Omega^{0,1} \otimes End(\mathcal{E}))$

$$CS(A) := \int_X Tr\left(\frac{A\bar{\partial}A}{2} + \frac{A^3}{3}\right) \wedge \Omega^{3,0})$$

The gauge group $Aut(\mathcal{E})$ acts on $\Gamma(X, \Omega^{0,1} \otimes End\, \mathcal{E})$

CS is invariant by identity component of gauge gp; however $\pi_0 Aut(\mathcal{E}) \xrightarrow{c} H_3(X, \mathbb{Z})$, value of CS gets shifted by $\int_{c(g)} \Omega^{3,0}$

($\Rightarrow$ think of CS as a multivalued function or as a closed 1-form)

crit pts of CS $\leftrightarrow$ holomorphic structures on $\mathcal{E}$.

★ More generally, $\mathcal{E}$ $\mathbb{Z}$-graded C^∞-bundle: then

$\{$superconnections$\}$ = loc. constant + $\Gamma(X, (\Omega^{0,\bullet} \otimes End\, \mathcal{E})^1)$

(NB: by construction, things are trivial on all but finitely many pieces).

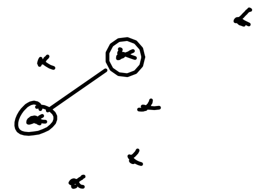
CS defines on this a holom. closed 1-form

$$\{\text{super-connections}\} \Big/ \begin{array}{l}\text{conn. component of}\\ \text{id in } Aut(\mathcal{E})\\ \text{(aut of } \mathbb{Z}\text{-graded bdle)}\end{array} \xrightarrow{CS} \mathbb{C}$$

If further quotient by $\pi_0 \operatorname{Aut}(\mathcal{E})$, get an $H_3(X,\mathbb{Z})$-cover & CS is def^{d} up to constant

* A stability condition yields a stack of finite type (moduli of semistable holom. structures)

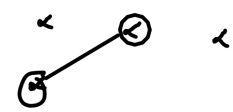
$\Leftrightarrow$ look at critical values of CS:



$$\underbrace{\left(\int \Omega^{3,0}\right)(H_3(X,\mathbb{Z}))}_{\text{periods of }\Omega} + \underbrace{\text{finite set}}_{\substack{\text{actual crit vals. up}\\ \text{to full gauge gp}}}$$

At each critical value, associate <u>critical cohomology</u>

* count of gradient flow lines of CS b/w critical values gives add^{l} data, <u>not</u> part of category-theoretic data.



(comes from count of traj^{s} on G_2-mfld $\mathbb{R}\times X$).

- There are now 2 types of wall-crossing effects:
    1) change of stab. condition
    2) change of complex structure on X

  both affecting count of s.s. objects.

<u>NB:</u> on A-model of mirror $X^\vee$:

ss-object count = count of special Lagrangians in given H_3-class.

gradient flow count = ?? perhaps count of coassociative trajectories in G_2 mfld $X^\vee\times\mathbb{R}$