M. Kontsevich – 21/1/10 – Mixed noncommutative motives

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periodic cyclic $HP^\bullet$ / k, char $k=0$ $\Longleftrightarrow$ de Rham cohomology

If have integral structure, then get Gauss-Manin connection on bundle of $HP^\bullet$ over moduli space!

In noncommutative case, how to do this?

Toën proposes K_{top} defined / $\mathbb{Z}$, $K_{top} \otimes \mathbb{C} \simeq HP$?

<u>Connes integration</u>:

A assoc algebra / $\mathbb{C}$, $A \to \mathrm{End}\,\mathcal{H}$ Hilbert space representation,
$F \in \mathrm{Aut}(\mathcal{H})$, $F^* = F$, $F^2 = 1$ involution, split $\mathcal{H}$ into ± 1 eigenspaces

Consider a Fredholm module: $\forall N \in \mathbb{N}$, $\forall a$, $\mathrm{Tr}\,|[a,F]|^N < \infty$.

$\Rightarrow$ define an "integration" functional on $HP^\bullet$

periodic cyclic chain $\sum_{\substack{n \geq 0 \\ \alpha}} a_0^\alpha \otimes \ldots \otimes a_{2n}^\alpha u^n$ (closed for diff! $b+uB$)
(even degree case)

$$\downarrow$$

$$\sum \frac{1}{(2\pi i)^n} \mathrm{Tr}\left(a_0^\alpha [F, a_1^\alpha] \cdots [F, a_{2n}^\alpha]\right)$$

• trace converges for $\forall n \geq \frac{N}{2}$.

This integration takes transcendental values.

<u>NB</u>: $A = A_0 \otimes \mathbb{C}$, A_0 def^d / $\mathbb{Q}$:
functional on $HP(A_0)$: $\mathbb{Q}$-vect space $\to \mathbb{C}$

• <u>Ex</u>: $A = C^\infty(X)$, spinor bundle, $F = \mathrm{sign}(D)$ $(= D . \sqrt{DD^*}^{-1})$
Dirac operator.

• <u>Ex</u>: residue: $\displaystyle\int_{|x|=1} \frac{dx}{1-P(x)} = \int a_0\, da_1$ $\qquad a_0 = \frac{1}{1-P(x)}$, $a_1 = x$
polynomial

Saturated dg-category $C \rightsquigarrow \dim HP^n < \infty$

$C = D^b\mathrm{Coh}(X)$ smooth proper $\rightsquigarrow HP^\bullet(C) = H^\bullet_{dR}(X \times \mathbb{P}^\infty)[u^{-1}]$
$k = \mathbb{C}$
module over $H^\bullet(\mathbb{P}^\infty) = \mathbb{Q}[[u]]$, β

Periodicity: $HP^{n+2} \simeq HP^n \otimes H^2(\mathbb{P}^1)$

$H^\bullet_{dR}(X \times \mathbb{P}^\infty)[u^{-1}]$ carries a pure Hodge structure.

<u>Q:</u> does HP^n have a pure Hodge structure of weight n, in the noncomm case?

<u>Mixed nc motives:</u>

- Ad hoc definition:

  1) Mot = category enriched over spectra
     objects = saturated dg-cats. over a given field k (char = 0)
     morphisms: $\mathrm{Hom}(C_1, C_2) :=$ (connective) alg. k-theory spectrum of $\mathrm{Fun}(C_1, C_2) = C_1^{op} \otimes C_2$
     This k-theory spectrum has structure of simplicial set:
     n-simplices = families of objects of $C_1^{op} \otimes C_2$ param^d by $\mathbb{A}^n$.

  2) Mixed nc motives = take triangulated envelope + Karoubi closure.

- <u>G. Tabuada</u>: <u>explanation:</u> a cohomology theory is

$$\begin{array}{c} (1,\infty)\text{-category of all small} \\ \text{triangulated Karoubi-closed dg-cat.} \end{array} \xrightarrow[\infty\text{-functor}]{\mathcal{H}} (1,\infty)\text{-triangulated cat.}$$

$$0 \longmapsto 0$$

st. • $\underset{\text{full subcat}}{A \hookrightarrow B} \to B/A \rightsquigarrow \mathcal{H}(A) \to \mathcal{H}(B) \to \mathcal{H}(B/A)$ exact triangle.

• commutes with filtered colimits

For <u>universal</u> cohomology theory $\mathcal{H}_{univ}$,

$$\mathrm{Hom}(\mathcal{H}(\hat{k}), \mathcal{H}(C)) = \text{noncomm. k-theory spectra.}$$

- <u>Ex:</u> $HP^\bullet$ has these properties
- mixed nc motives as defined above is a subcat. of $\mathcal{H}_{univ}$ (in particular, restrict ourselves to saturated dg-cats)

Rmk: Mixed nc motives $\xleftrightarrow[\text{full}]{}$ Voevodsky

- Objects in mixed nc motives:

  - Twisted complexes [note: K-theory connective i.e. only in cohom. degrees ≤ 0]

    $$0 \to \dots \to C^i \xrightarrow{F^i} C^{i+1} \xrightarrow{F^{i+1}} C^{i+2} \to \dots 0 \quad (\text{deform}^n \text{ of } \oplus C^i[-i])$$

    C^i saturated dg cat's

    $F^i : C^i \to C^{i+1}$ functors

    $F^{i+1} \circ F^i \underset{\mathbb{A}^1\text{-hom.}}{\sim} 0$ $\quad$ $\mathbb{A}^1$-homotopies

    $\mathbb{A}^2$-homotopies b/w these $\mathbb{A}^1$-homotopies, and so on.

- Rmk: Finite diagram of dg-cats & functors, strictly associative:

  simplicial set S, $\quad \alpha \in S_0$ vertex $\mapsto C_\alpha$

  edges $\longmapsto$ functors

  simplices $\longmapsto$ homotopies ...

  It seems natural to consider homotopy colimits of such diagrams rather than arbitrary twisted complexes. Would these be good enough?

- Twisted complex $\dots \to C^i \xrightarrow{F^i} C^{i+1} \to \dots$ $\longmapsto$ $\oplus\, CC^-(C^i)[i]$ negative cyclic complex with total differential

  $\Rightarrow HC^-(\dots)$

  Conj: HC^- is a free module over $k[[u]] = H_{dR}(\mathbb{P}^\infty)$

  ($\Leftrightarrow$ degeneration of Hodge-to-de Rham spectral sequence)

NB: $HC^-(\dots)$ carries all the structures and filtrations expected of Hodge structure, e.g. $\begin{cases} \text{filtration from } CC^- \\ \text{filtration from twisted complex (truncate to } \bigoplus_{i \geq i_0} C^i) \end{cases}$

Ex: X smooth proper $\supset D_1, D_2$ smooth divisors, $D_1 \cap D_2 = \emptyset$

$$\Rightarrow \quad \underset{-1}{D^b\mathrm{Coh}(D_1)} \xrightarrow{(i_1)_*} \underset{0}{D^b\mathrm{Coh}(X)} \xrightarrow{(i_2)^*} \underset{1}{D^b\mathrm{Coh}(D_2)}$$

get mixed Hodge structure on $H^*(X - D_1, D_2)$.

(baby example: $X = \mathbb{P}^1$, $D_1 = 2$ pts, $D_2 = 2$ pts).

Should extend to: D_1, D_2 normal crossings.

- Object $\bullet \in \mathcal{C}$ saturated cat.

$\Rightarrow$ form twisted complex $\cdots \to 0 \to \bullet \to \mathcal{C} \to 0 \to \cdots$

$$\Rightarrow \quad K_0^{deg}(C) \longrightarrow \pi_0 \left[\underset{\longleftarrow}{\mathrm{holim}} \left(\begin{array}{ccc} & & HC^-(C) \\ & & \downarrow \\ K_{top}(C) & \longrightarrow & HP(C) \end{array} \right) \right]$$

$$0 \to HP^1(C) \Big/ F^1 HP^1 + K^1_{top} \longrightarrow H^0_{Deligne}(C) \to \underbrace{K^0_{top} \cap F^0 HP^0}_{\text{Hodge classes}}$$

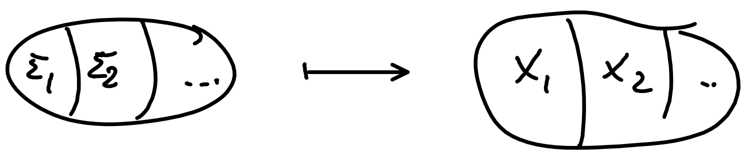
$$\cong$$

$$\underbrace{H^{odd}_{dR}(X) \Big/ \bigoplus_{p \geq q} H^{p,q} + H^q_B(X, \mathbb{Z})}_{\text{torus of intermediate Jacobian}}$$

- <u>On A-model:</u> $X_i \quad X_{i+1} \quad \cdots$ symplectic mflds (singular)

$X_i = \bigcup X_{i,\alpha}$, $X_{i,\alpha}$ sympl. of some dimension

$\leadsto$ holom. curves Σ where $\Sigma = \bigsqcup \Sigma_i$, $\Sigma_i \to X_i$ holom.

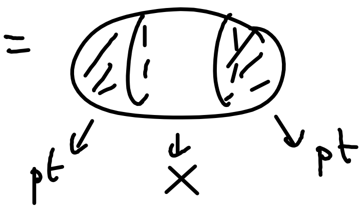


cf. Wehrheim-Woodward

<u>Rmk:</u> this is how a twisted complex of saturated categories naturally comes up in the A-model!

$$\cdots \to \mathcal{F}(X_i) \xrightarrow{L_i} \mathcal{F}(X_{i+1}) \xrightarrow{L_{i+1}} \cdots$$

Ex: $L_1, L_2 \subset X$, $L_1 \cap L_2 = \emptyset$, build. $pt \to X \to pt$,

$\Sigma = \mathbb{P}^1 =$ 

$\Longleftrightarrow$ cylinders with $\partial C \subset L_1 \cup L_2$.

Multiple cover contribution:

$$\sum \frac{1}{n} \exp(-(\text{Area of cyl.})\, n) = \log\left(1 + e^{-\text{Area}}\right).$$

log suggests there should be a mixed H. structure.

(Rmk: $(X^\vee, D_1 \cup \dots \cup D_n)$ $\overset{\text{mirror}}{\sim}$ $X \downarrow \mathbb{C}^n$ (n superpotentials)

$\sum D_i = -K_X$ $\qquad H^\circ(X) \simeq H^\circ(X^\vee - \cup D_i)$

mixed HS on $H^\circ(X^\vee \setminus \cup D_i)$

extend over $(\mathbb{P}^1)^n$

monodromy at ∞ .... induces filtrations

$\rightsquigarrow$ mixed Hodge structures on H°)

[works out diagrams of categories & functors for restrictions to D_i & their intersec^{ns}.].