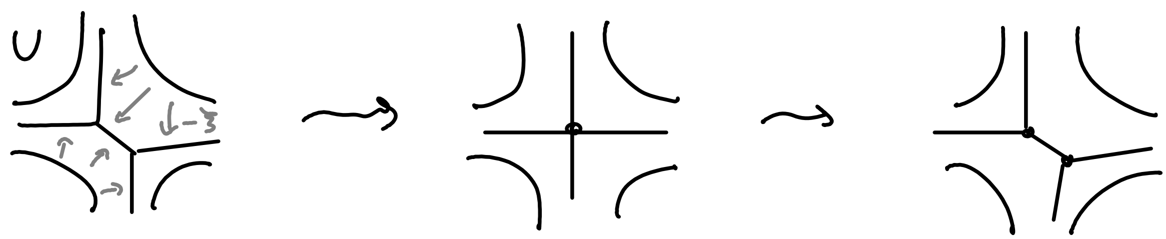
M. Kontsevich – 20/1/09 – Singular Lagrangian branes

Singular Lagrangian $L \subset (X,\omega)$ $\rightsquigarrow$ an ε-neighborhood $\mathcal{U} \supset L$ is a <u>Liouville mfld</u>

Def: Liouville mfld $(\mathcal{U},\omega)$ = compact sympl. mfld with boundary, st $\exists$ vector field ξ, $L_\xi \omega = \omega$ $\quad(\Leftrightarrow \omega = d\iota_\xi \omega)$, ξ outwards at $\partial\mathcal{U}$.

Running flow of ξ backwards collapses $\mathcal{U}$ onto a singular Lagr. skeleton.

<u>NB</u>: set of choices of ξ is contractible if nonempty.
Deforming choice of ξ can modify the structure of the spine



- Say singularities of L are <u>good</u> if nbd(L) is Liouville and has Liouville v.f. whose spine $\cong L$. Don't know what good sings. look like, but seems reasonable to expect L Whitney stratified.

- On L with good stratification, $\exists$ natural <u>cosheaf</u> Φ_L in homotopy sense of finite type dg-categories / $\mathbb{Z}$.

Def: given a finite simplicial set S, (oriented, ordered ...), a <u>diagram of dg-cat / S</u> is:
- $\forall$ vertex $s \in S_0$, C_s dg-category
- $\forall$ edge $\underset{s_0}{\bullet} \longrightarrow \underset{s_1}{\bullet}$, dg functor $F_{s_0 s_1} : C_{s_0} \to C_{s_1}$
- simplex (triangle s_0, s_1, s_2) $\Rightarrow$ quasi isomorphism $F_{s_1 s_2} \circ F_{s_0 s_1} \xrightarrow{\sim} F_{s_0 s_2}$

...

- We'll consider situations where each dg cat. C_s is given by a finite quiver Q, whose arrows are ordered $(\alpha_1, \alpha_2, \dots)$, grading $\in \mathbb{Z}$ or $\mathbb{Z}/2$

$\rightsquigarrow$ C_s = path categories, $d(\alpha_i) \in \text{Path}(\alpha_{<i})$.

path subcat. of previous arrows.

The homotopy limit assoc. to simplicial set S can be understood by an explicit model.

Eg. when Q = just a single vertex, <u>diagram of points</u> $\Leftrightarrow$

- dg-cat. with objects $= S_0$
- <u>mor</u> = alg. freely generated by non-degenerate simplices of dim >0

  ordered n-simplex $(i_0 \dots i_n)$ $(n \geq 1)$ $\rightsquigarrow$ gives $\alpha_{i_0 \dots i_n} : i_0 \longrightarrow i_n$

  of degree $1-n$.
- <u>differential</u> $d\alpha_{i_0 \dots i_n} = \sum_{j=1}^{n-1} (-1)^{j-1} \left(\alpha_{i_0 \dots i_j} \alpha_{i_j \dots i_n} + \alpha_{i_0 \dots \hat{i_j} \dots i_n} \right)$
- $\forall (i_0, i_1)$, postulate α_{i_0, i_1} is an <u>isomorphism</u>

  ie. make $\text{Cone}(\alpha_{i_0, i_1}) \simeq 0$.

  Do this by adding extra generators = nullhomotopies for these cones.

$\Rightarrow$ End up with essentially <u>dg-path groupoid</u> of the simplicial set, $C_*(\Omega |S|)$, ie. objects = vertices of simplicial complex $|S|$,

$\text{hom}(v_0, v_1) \sim C_*(\Omega_{v_0, v_1} |S|)$

path space.

- Can similarly understand case where we attach more complicated quivers to the vertices of S.

- If L is compact, $\Phi_L(L)$ (ie. global sections of the cosheaf) should be a $\mathbb{Z}/2$-graded dg-category of finite type, with a Calabi-Yau structure.

- $\mathcal{C}$ smooth dg-cat., $\mathcal{C} = \mathrm{Perf}(A)$ Definition of CY category:
element in $HC^-(\mathcal{C})$ Hochschild complex

$$\downarrow$$

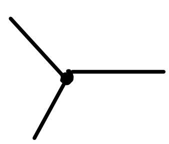
$HH(\mathcal{C}) = \mathrm{RHom}(A^\vee, A)$ should be an isom. of A-bimodules

$\left(A^\vee = \mathrm{RHom}_{A\text{-mod-}A}(A, A \otimes A)\right)$.

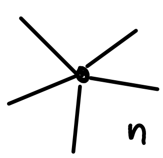
- L smooth, oriented, with spin structure; pick simplicial decomp. of L.
Then $\Phi_L(L) \simeq \mathrm{Chains}(\Omega(L, x_0)) = \underrightarrow{\mathrm{holim}}$ (diagram of pts, twisted by $w_1(L)$ & $w_2(L)$ Stiefel-Whitney classes.)

Twisting by Stiefel-Whitney classes:
given any diagram of dg-cat/S. (w/ $\mathbb{Z}/2$-coeffts)
can twist by elts of $H^1(|S_\cdot|, \mathbb{Z}/2)$ (where $\mathbb{Z}/2$ acts on $\mathbb{Z}/2$-dg cat. by shifts).
$H^2(|S_\cdot|, \mathbb{Z}/2)$ ($\mathbb{Z}/2$ on functors ...).

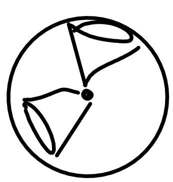
- What about singularities?



$\rightsquigarrow$ stalk at vertex = rep$^{\underline{ns}}$ of A_2 $\bullet \to \bullet$

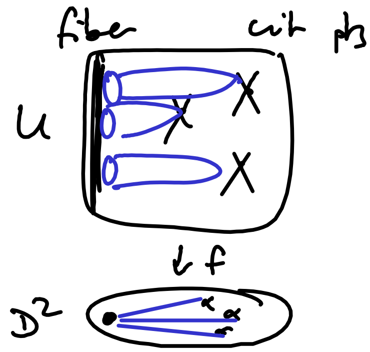


n edges $\rightsquigarrow$ stalk at vertex = A_{n-1}.



cone over Legendrian in small ball $\rightsquigarrow$ stalk = Fukaya-Seidel cat. (look at Reeb flow on sphere).

$\star$ $\Phi_L(L)$ for compact L : $\mathcal{U} \supset L$ Liouville neighborhood,
set it up as total space of a Lefschetz fibration (up to modif$^{\underline{n}}$)
$\mathcal{U}' = f^{-1}$(bounded set), $f : \mathcal{U}' \to$ disc Lefschetz fibration



- To the fiber we associate $\phi(U')$ finite type CY dg-cat., with spherical objects $\mathcal{E}_1 \dots \mathcal{E}_k$ ($\sim$ vanishing cycles).
- FS-cat. of (U, f): is saturated, with exceptional collection $\tilde{\mathcal{E}}_1, \dots, \tilde{\mathcal{E}}_k$

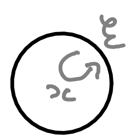
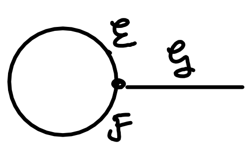
$$RHom(\tilde{\mathcal{E}}_i, \tilde{\mathcal{E}}_j) = \begin{cases} 0 & i > j \\ id & \\ RHom_{\phi(U')}(\mathcal{E}_i, \mathcal{E}_j) & \end{cases}$$

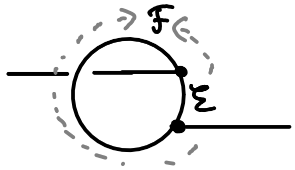
- FS carries a natural transformation $\text{Serre} \to Id\left[\frac{\dim}{2}\right]$

  Localize wrt it, i.e. kill image of $\text{Cone}(\text{Serre} \to Id)$ yields <u>wrapped category</u> $\rightsquigarrow$ this is $\phi_L(L)$.

[<u>In other terms</u>: for any compact L, we want to associate $\phi_L(L) :=$ wrapped Fukaya cat. of U = Liouville nbd. of L.].

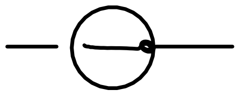
<u>More examples:</u>

L	$\phi(L)$	$\simeq D^b Coh$ for
point $\bullet$ / $\mathbb{R}^n$ ———	$\mathbb{Z}$-mod.	point
	$\mathbb{Z}[x^{\pm 1}]$-modules	$\mathbb{A}^1 \setminus \{0\}$
	(namely: over interval $\underline{\quad}$ $\mathcal{E} \in \mathbb{Z}$-mod glue at end pts $\Leftrightarrow$ $x \in End(\mathcal{E})$ invertible),	
	$\mathbb{Z}[x]$-modules	$\mathbb{A}^1$
	(exact triangle $\mathcal{E} \to \mathcal{F} \to \mathcal{G}$ + map $x : \mathcal{E} \to \mathcal{F}$, $\mathcal{G} = \text{Cone}$)	

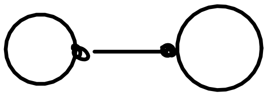


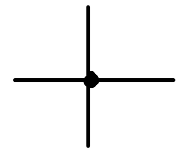
 - modules

$\mathbb{P}^1$

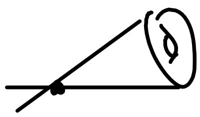


$\mathbb{P}^1$



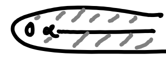


$\mathbb{A}^1 \cup_0 \mathbb{A}^1$
$= \{xy = 0\}$ in $\mathbb{A}^2$



$L \subset \mathbb{R}^6$
cone over T^2 $\longleftrightarrow$ $\mathbb{P}^1 \setminus \{0, 1, \infty\}$

$(|x| = |y| = |z|, \ xyz \in \mathbb{R}_{\geq 0})$

(NB: L is Liouville skeleton for preimage of  $\mathbb{R}_+$ in $\mathbb{C}^3 \xrightarrow[W = xyz]{} \mathbb{C}$)

X compact symplectic manifold, $[\omega] = c_1(\mathcal{L})$,
$X \supset D$ for D zeroes of section of $\mathcal{L}^{\otimes k}$, $k \geq 1$ (Donaldson hyp.)
$\rightsquigarrow$ $X - \mathcal{V}(D)$ is Liouville, can be contracted to a spine L

$\Phi(L)$ finite type CY dg category

Seidel: Fukaya cat. of X = deformation of $\Phi(L)$, hence given by a solution of Maurer-Cartan ($\rightsquigarrow$ dg-algebra)

$$\in C^{cycl}(\Phi(L)) \hat{\otimes} q\mathbb{Q}[[q]]$$

(counts boundaries of holomorphic discs in (X, L)).
= naturally live on cyclic chains of $C_*(\Omega L)$.

To get GW invts of X from this:

Claim: if $\mathcal{C}$ smooth dg-cat, CY, then get

$$HH(\mathcal{C})^{\otimes n} \otimes H_*(\mathcal{M}_{g, \vec{n} + \vec{m}}) \longrightarrow HH(\mathcal{C})^{\otimes m}$$

$n \geq 0$, $m \geq 1$, $g \geq 0$ $\quad \dots \rightarrow$ GW invts