M. Abouzaid – 23/1/10 – Floer homology and string topology

$M = T^*Q$, Q smooth compact mfld

<u>Thm:</u> $\exists^*$ fully faithful embedding $\mathcal{W}(T^*Q) \longrightarrow \mathrm{mod}(C_{-*}(\Omega_q Q))$ whose image agrees with the triangulated closure of the free module.

(* fine print: the functor always exists, but proof of fully faithful currently assumes univ. cover $\tilde{Q}$ has finite homotopy type).

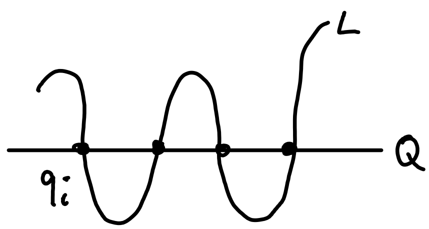
<u>Csq:</u>
- T_q^*Q cotangent fiber generates the wrapped Fukaya category
- every exact Lagrangian in T^*Q (conical at ∞) has a "filtration" by cotangent fiber.

<u>Define:</u> <u>path category</u> $\mathcal{P}(Q) = \begin{cases} \text{- objects} = q \in Q \\ \text{- morphisms: } \mathrm{Hom}(p,q) = C_{-*}(\Omega_{p,q}Q) \\ \text{- composition} = \text{concatenation} \end{cases}$

<u>Prop:</u> Whenever $Q \hookrightarrow M$ Lagr., M Liouville, $\exists$ functor $\mathcal{W}(M) \longrightarrow \mathrm{Tw}\,\mathcal{P}(Q)$

(Twisted complexes: $(X_i, D = (\delta_{ij}))$, $\delta_{ij} = 0$ if $i \geq j$, $\deg \delta_{ij} = 1$, $\partial D \pm D^2 = 0$.)

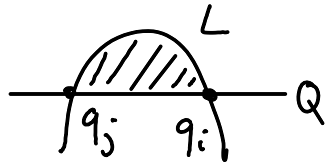
1) <u>Construction:</u>



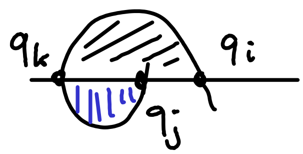
given $L \in \mathcal{W}(M) \rightsquigarrow$ build a twisted complex $\mathcal{F}(L) \in \mathrm{Tw}\,\mathcal{P}(Q)$ as follows:

- complex is built from the intersection points $q_i \in Q \cap L$
  ordering = by action
  (key point: if $\mathcal{A}(q_i) < \mathcal{A}(q_j)$ then moduli space of holom. strips between L & Q from q_j to q_i is <u>empty</u> since energy < 0).

• For each pair, consider $\bar{\mathcal{M}}(q_i, q_j) = \left\{ \begin{array}{l} \text{compactified moduli space} \\ \text{of hol. strips } q_i \to q_j \end{array} \right\}$



This is a mfld with boundary, and the ∂ is covered by codim. 1 strata = images of $\bar{\mathcal{M}}(q_i, q_k) \times \bar{\mathcal{M}}(q_k, q_j) \longrightarrow \bar{\mathcal{M}}(q_i, q_j)$



• $\partial[\bar{\mathcal{M}}(q_i, q_j)] = \sum_{q_k} \pm [\bar{\mathcal{M}}(q_i, q_k)] \times [\bar{\mathcal{M}}(q_k, q_j)]$ $\quad (*)$

+ $\exists$ evaluation map $ev : \bar{\mathcal{M}}(q_i, q_j) \longrightarrow \Omega_{q_i, q_j} Q$
(ev. at boundary)

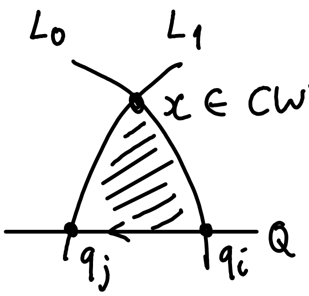
<u>Define:</u> $\delta_{ij} = ev_*[\bar{\mathcal{M}}(q_i, q_j)]$

$(*) \Rightarrow$ Maurer-Cartan eqn for (δ_{ij}), ie. we have a twisted complex.

<u>NB:</u> representing L in this manner makes it a twisted complex built out of the cotangent fibers $T^*_{q_i} Q$ (note $\mathcal{F}(T^*_{q_i} Q) = q_i$) with filtration given by ordering the q_i by action.

2) • On <u>morphisms</u>: given L_0, L_1, want map
$$CW^*(L_0, L_1) \longrightarrow \underline{\mathrm{Hom}}(\mathcal{F}(L_0), \mathcal{F}(L_1))$$

Recall $\mathrm{Hom}((T_1, \delta^1_{ij}), (T_2, \delta^2_{ij})) = \left(\bigoplus_{i,j} \mathrm{Hom}(X^1_i, X^2_j),\ \partial \pm - \circ D_1 \pm D_2 \circ - \right)$
where $T_1 = \bigoplus X^1_i$, $T_2 = \bigoplus X^2_j$



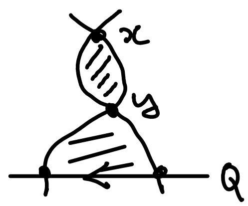
$x \in CW^*(L_0, L_1)$ (x = either an actual intersection or a Reeb chord)

$\mathcal{M}(q_j, q_i, x)$ = holom. maps from a "triangle" to M with corners at x, q_i, q_j

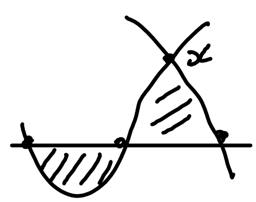
Evaluation map: $\mathcal{M}(q_j, q_i, x) \longrightarrow \Omega_{q_i q_j} Q$

choose $[\mathcal{M}(q_i, q_j, x)]$, and define $\mathcal{F}^1(x) = \bigoplus_{i,j} ev_*[\mathcal{M}(q_j, q_i, x)]$.

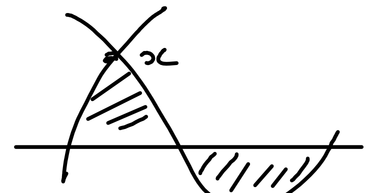
This satisfies expected properties since $\partial\mathcal{M}(q_j, q_i, x)$ consists of:



$\mathcal{F}^1 \circ \mu^1_{CW^*}(x)$,

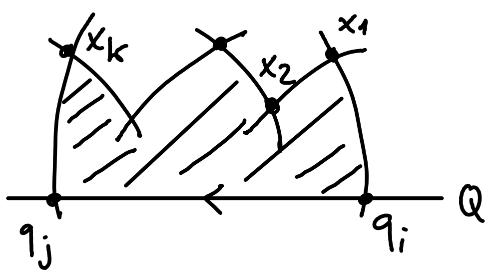


$D^1 \circ \mathcal{F}^1(x)$,



$\mathcal{F}^1(x) \circ D^0$

- Higher maps in the A_∞-functor are built similarly:



gives (i,j) component of $\mathcal{F}^k(x_1, \dots, x_k)$.

▲

Let's return to the case of $M = T^*Q$. We need to:

1) prove that $\mathcal{F}$ induces an <u>A_∞ quasi isom.</u> $CW^*(T_q^*Q, T_q^*Q) \to C_{-*}(\Omega_q Q)$

2) prove that T_q^*Q generates $\mathcal{W}$.

- <u>Part 1</u> is where the stupid restriction on homotopy type comes up.

Abbondandolo - Schwarz build a map $C_{-*}(\Omega_q Q) \xrightarrow{AS} CW^*(T_q^*Q)$ inducing an isom. on homology. (not extended to A_∞-map)

Prove the map given by $\mathcal{F}$ is a 1-sided inverse of AS.

- <u>Part 2</u>: <u>Proof of generation</u>

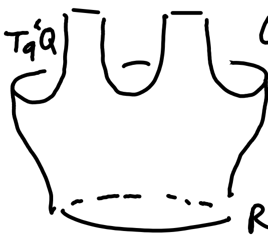
Recall criterion: | if $HH_*(CW^*(T_q^*Q)) \to SH^*(T^*Q)$ hits the identity then T_q^*Q <u>split-generates.</u>

Set up a diagram:

$$\begin{tikzcd}
HH_*(CW^*(T_q^*Q)) \arrow[r, "\text{yesterday}"] \arrow[d, "\simeq \text{ Part 1}"', "\mathcal{F}_*"] & SH^*(T^*Q) \arrow[d] \\
HH_*(C_{-*}(\Omega_q Q)) \arrow[r, "\simeq"', "\text{isom. by Goodwillie}"] & H_{-*}(\mathcal{L}Q)
\end{tikzcd}$$

Map at top edge = (cf yesterday)

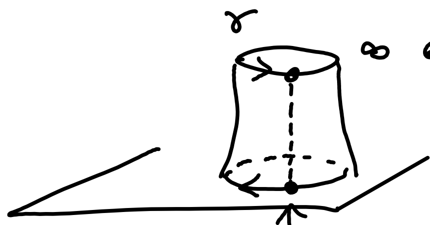
annulus with ∞ modulus



Reeb chords on T_q^*Q (+ marker) ($\in$ cyclic bar$(CW^*(T_q^*Q))$)

(parametrized) Reeb orbit in T^*Q. ($\in SC^*(T^*Q)$)

Map at right edge := punctured holom. discs w/ boundary on Q:



∞ end = Reeb orbit (parametrized) $\in SC^*(T^*Q)$

free loop in Q

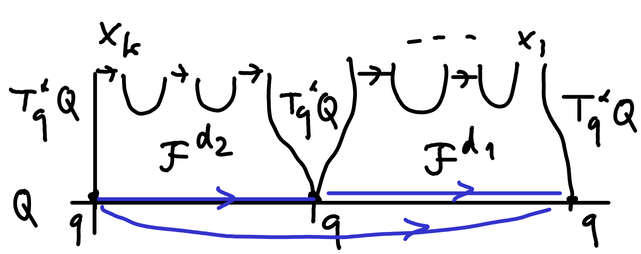
gives evaluation map $ev_*[\mathcal{M}(\gamma)] \in C_{-*}(\mathcal{L}Q)$.

* To show the diagram commutes, look at compositions:

top & right edges: glue together & deform domain => composition counts

finite modulus annuli w/ ∂ on Q and on T_q^*Q.

left & bottom edges:



Take tensor product of A_∞-maps $CW^* \xrightarrow[\mathcal{F}]{} C_{-*}(T_q^*Q)$ constructed above, to get elt of cyclic bar complex of $C_{-*}(\Omega_q Q)$

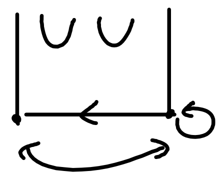
Then concatenate paths $q \rightsquigarrow q$ to get closed loops

(This is the map $HH_\alpha(C_{-\alpha}(\Omega_q Q)) \to H_\alpha(\mathcal{L}Q)$)

(Note: $\mathcal{F}: \mathcal{A} \to \mathcal{B}$ A_∞-hom. $\Rightarrow$ induced map on cyclic bar complex is $a_n \otimes \dots \otimes a_1 \mapsto \sum \mathcal{F}^{d_1}(\dots) \otimes \dots \otimes \mathcal{F}^{d_k}(\dots)$)

This again corresponds to (degenerate) annuli w/ boundary on $T_q^* Q$ & Q

The length 1 case =



same point.

* Thus: diagram commutes [up to homotopy?] and since lower-left side is quasi isom., upper map must hit identity in SH^α.

Hence $T_q^\alpha Q$ split generates $\mathcal{W}(T^* Q)$.

But we have a functor to $Tw(\mathcal{P}(Q))$

$\cong$ triangulated closure of $T_q^* Q$.

Hence it's actually generated (no need to take split-closure).