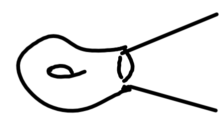
M. Abouzaid – 22/1/10 – A generating criterion for Fukaya categories

The Wrapped Fukaya category is defined for Liouville manifolds
(ie. Stein manifolds & affine varieties)

Def: Liouville mfld = exact sympl. mfld, convex at ∞
ie. $(M, \omega = d\lambda)$, st. Liouville field $X_\lambda :=$ v.f. st $\omega(X_\lambda, -) = \lambda$
points outwards along ∂M.

If M = Riem. surface, convexity condition can always be achieved.

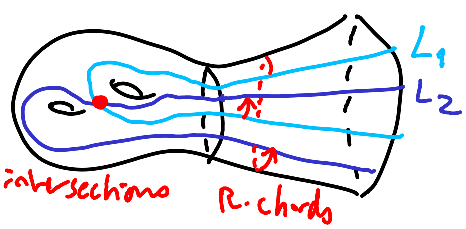
Convexity allows us to attach infinite cone & have max. principle arguments to ensure compactness for J-hol. curves.

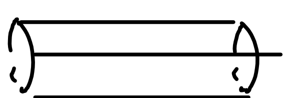


Objects of $\mathcal{W}(M) :=$ exact Lagrangians in M $(\lambda_{|L} = df)$
st. $f_{|\partial L}$ is locally constant (ie. ∂L legendrian)

Morphisms: $CW^*(L_1, L_2) := \mathbb{Z}[L_1 \cap L_2] \oplus \mathbb{Z}[\text{Reeb chords } \partial L_1 \to \partial L_2]$
wrapped Floer complex
(in general, ∞-dim!)

(this is not symmetric: not a CY category)



Ex:  L $HW(L, L) \cong \mathbb{Z}[u, u^{-1}]$.

Symplectic cohomology: (analogue of quantum cohomology in the noncompact setting)

If M is Liouville, $\exists$ well def$^{\underline{d}}$ notion of "quadratic at ∞"

Eg: $M = T^*Q$, (p, q) coordinates, $H = |p|^2$

Function $H \rightsquigarrow$ Floer homology $SH^*(M)$

Complex $SC^*(M)$ gen^d by 1) critical points in the interior $(\sim H^*(M))$
2) orbits of the Reeb flow

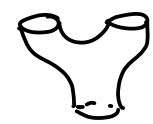
<u>Eg:</u> $SH^*(T^*Q) \cong H_{n-*}(\mathcal{L}Q)$, in manner compat. w/ S^1-action.

<u>Question:</u> given a collection of Lags. in $\mathrm{Ob}(\mathcal{W})$, when do they generate $\mathcal{W}$?

Consider a full subcategory $\mathcal{B} \subset \mathcal{W}$ with objects L_i

<u>Thm:</u> $\exists$ natural map $HH_*(\mathcal{W},\mathcal{W}) \longrightarrow SH^*(M)$. If the identity element in $SH^*(M)$ lies in the image of $HH_*(\mathcal{B},\mathcal{B}) \longrightarrow HH_*(\mathcal{W},\mathcal{W}) \longrightarrow SH^*(M)$ then $\mathcal{B}$ split-generates $\mathcal{W}$.

<u>NB:</u>

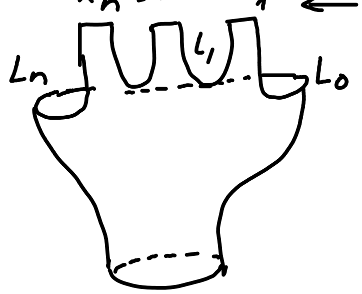
- $SH^*(M)$ is a ring: product comes from 
  but id. comes from <u>ring map</u> $H^*(M) \to SH^*(M)$
- expect: if assumption holds then map $HH_*(\mathcal{W},\mathcal{W}) \to SH^*(M)$ is an isomorphism. Should follow from the Thm.

<u>What's the map?</u>

Use cyclic bar complex for HH_*, ie.
generators of $CC_* \rightsquigarrow x_n \otimes \dots \otimes x_1 \in CW^*(L_{n-1}, L_n) \otimes \dots \otimes CW^*(L_0, L_1)$.

$x_n \cdots x_1 \leftarrow$ Reeb chords between the $L_i \in CW^*$

The map counts

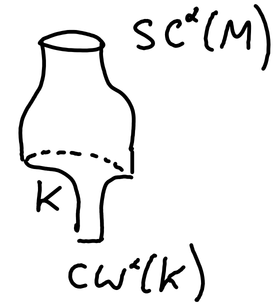


$\gamma \leftarrow$ Reeb orbit $\in SC^*$

- Also, $\exists$ ring map $SH^*(M) \longrightarrow HH^*(\mathcal{W},\mathcal{W})$

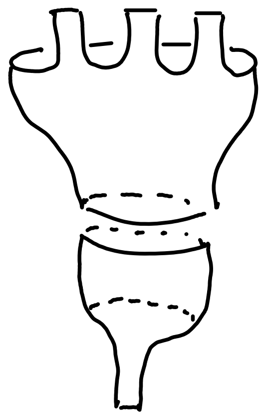
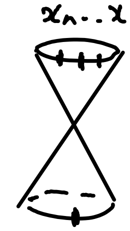
Projecting to a single object,

$$\begin{tikzcd} SH^*(M) \arrow[r] \arrow[dr] & HH^*(W,W) \arrow[d] \\ & HW^*(K,K) \end{tikzcd}$$

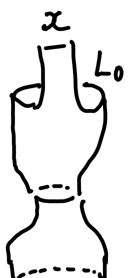
given by 

Composing, we get a map $HH_*(\mathcal{B},\mathcal{B}) \to HW^*(K,K)$ $\forall K$, and assumption of thm $\Rightarrow$ id_K is in the image of this map.

Idea pf. thm: (cf. ideas of Fukaya & Biran-Cornea)

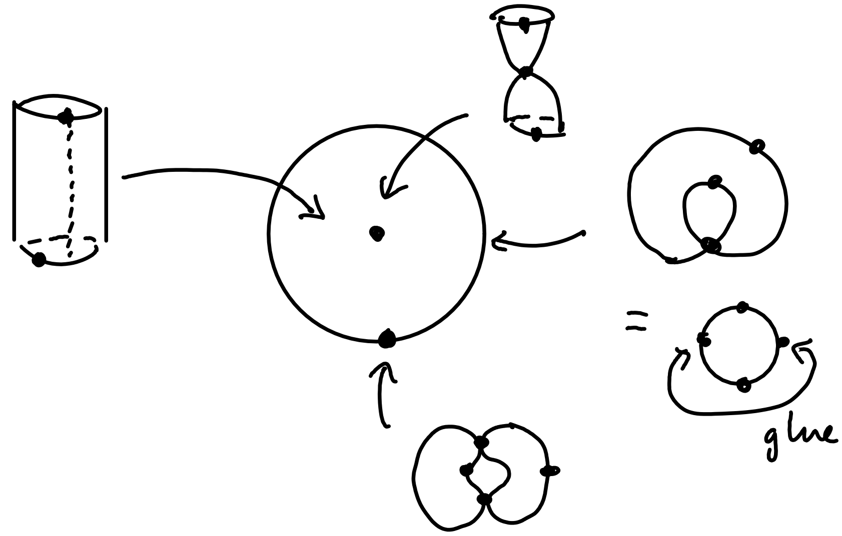
Compose the 2 maps:  = singular annulus 

Cardy relation: the simplest instance, when only 1 input:

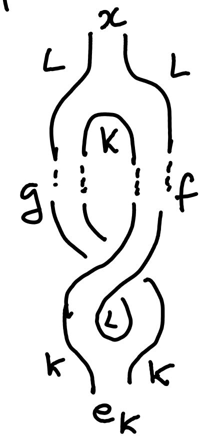


map sends $x \mapsto e_K$ regardless of chosen moduli param. for domain annulus.

Moduli space of annuli w/ 2 ∂ marked points



At special ∂ point:



i.e. $\exists\, f \in HF^*(L,K)$, $g \in HF^*(K,L)$ st.

$$\mu_2(g,f) = e_K$$

Now, $Y \underset{g}{\overset{f}{\rightleftarrows}} X$, $\mu_2(g,f) = id_Y \Rightarrow Y$ is a summand of X.

<u>In general case:</u>

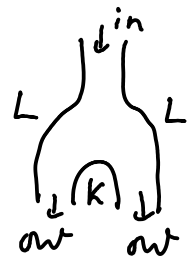
$\mathcal{Y}^l(K) = \oplus\, CW^*(K, L_i)$ left Yoneda module over $\mathcal{B}$

$\mathcal{Y}^r(K) = \oplus\, CW^*(L_i, K)$ right ——"——

$\exists$ map of A_∞-bimodules / $\mathcal{B}$, $\mathcal{B} \xrightarrow{\Delta} \mathcal{Y}^l(K) \otimes \mathcal{Y}^r(K)$

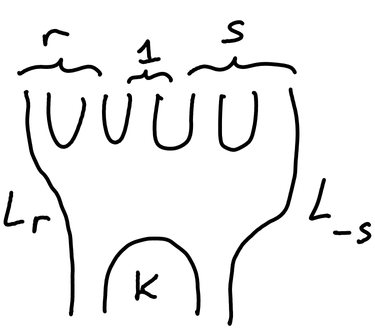
namely, maps $\Delta^{r|1|s}$ $\forall\, r,s \geq 0$:

linear part $\Delta^{0|1|0}$:



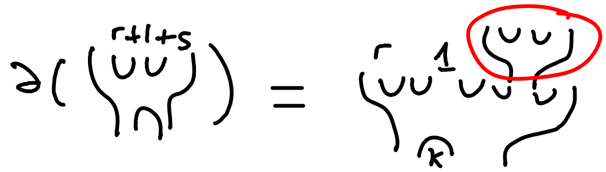
$$\begin{array}{c} HW^*(L,L) \\ \downarrow \\ HW^*(K,L) \otimes HW^*(L,K) \end{array}$$

More generally, $\Delta^{r|1|s}$:



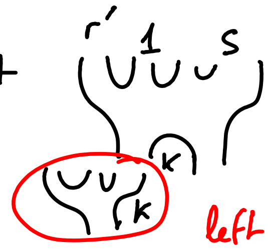
Check this is a bimodule map by looking at degenerations.

i.e. ∂ of 1-dim'l moduli spaces:



differential in bar complex of $\mathcal{B}$

\+ same on other side



left action of $\mathcal{B}$ on $\mathcal{Y}^\ell(K)$

\+ same on other side (right action of $\mathcal{B}$ on $\mathcal{Y}^r$)

- Given map of A_∞-bimodules, get an induced map on HH_*:

$$\begin{tikzcd}
HH_*(\mathcal{B},\mathcal{B}) \arrow[r] \arrow[d, "HH_*(\Delta)"] & SH^*(M) \arrow[d] \\
HH_*(\mathcal{B}, \mathcal{Y}^\ell(K) \otimes_K \mathcal{Y}^r(K)) \arrow[r] & HW^*(K,K) \\
\mathcal{Y}^r(K) \otimes_{\mathcal{B}} \mathcal{Y}^\ell(K) \arrow[u, phantom, "\cong" rotate=90]
\end{tikzcd}$$

By assumption, $\exists\, \sigma \longmapsto \mathrm{id}_M$

$$\begin{tikzcd}
\sigma \arrow[r, mapsto] \arrow[dd, mapsto, bend left] & \mathrm{id}_M \arrow[dd, mapsto] \\[-1em]
HH_*(\mathcal{B},\mathcal{B}) \arrow[r] \arrow[d, "HH_*(\Delta)"] & SH^*(M) \arrow[d] \\
\mathcal{Y}^r(K) \otimes_{\mathcal{B}} \mathcal{Y}^\ell(K) \ni \tilde{\sigma} \arrow[r] & HW^*(K,K) \ni \mathrm{id}_K
\end{tikzcd}$$

$\tilde{\sigma} := HH_*(\Delta)(\sigma) \longmapsto \mathrm{id}_K$

This implies that K is split gen^d by $\mathcal{B}$

(easier case: if $\mathrm{id}_K \in \mathrm{image}\,(HW^*(K,L) \otimes HW^*(L,K) \to HW^*(K,K))$
then K direct summand in L)

The above works fine over $\mathbb{Z}$ thanks to exactness.

- What about compact symplectic mflds? (AFOOO, in progress)

  $\to$ need to switch from $\mathbb{Z}$ to Novikov ring over $\mathbb{R}$ ie. $\{\sum a_i t^{\lambda_i}\}$, $\lambda_i \in \mathbb{R}$

<u>Fukaya:</u> The Fukaya cat. of a compact sympl. mfld is a cyclic A_∞-category over $\Lambda_{\mathbb{R}}$.

A weak consequence is: given $\mathcal{B} \subset \mathcal{F}uk(M)$ full subcat.,

Poincaré duality $\Rightarrow$ $\begin{cases} \bullet\ \mathcal{B}^\vee \simeq \mathcal{B}[n] & \text{as an } A_\infty\ \mathcal{B}\text{-bimodule} \\ \bullet\ \mathcal{Y}_l(K)^\vee \simeq \mathcal{Y}_r(K)[n] & \text{as } A_\infty\text{-modules} \end{cases}$

Also, $\exists$ evaluation map $\mathcal{Y}_l(K) \otimes_k \mathcal{Y}_r(K) \xrightarrow{\mu} \mathcal{B}$.

Dualizing, this gives:

$$\begin{array}{ccc} \mathcal{B}^\vee & \xrightarrow{\mu^\vee} & \mathcal{Y}_r^\vee \otimes \mathcal{Y}_l^\vee \\ \wr\wr & & \wr\wr \\ \mathcal{B}[n] & \longrightarrow & \mathcal{Y}_l \otimes \mathcal{Y}_r[2n] \end{array}$$

This is the map we want, ie. take $\underline{\underline{\Delta = \mu^*}}$.

We now have:

$$\begin{tikzcd} HH_*(\mathcal{B},\mathcal{B}) \arrow[r, "\rho", "\text{contr. by FOOO}"'] \arrow[d, "HH_*(\Delta)"'] & QH^*(M) \arrow[d] \\ HH_*(\mathcal{Y}_l \otimes \mathcal{Y}_r) \simeq \mathcal{Y}^r(K) \otimes_{\mathcal{B}} \mathcal{Y}^l(K) \arrow[r, "\mu"] & HF^*(K,K) \end{tikzcd}$$

and argument works in the same way if $id \in im(\rho)$.

- The unit for $QH^*(M)$ is the fundamental class $[M]$.

  Hence the generation criterion $id \in im(\rho)$ is:
  $\exists L_1 \ldots L_n$, $x_i \in L_{i-1} \cap L_i$, st. moduli space of discs with ∂ on L_i and corners at x_i + 1 interior marked point has $ev_* \mathcal{M} = [M]$.

<u>Consequence</u>: $\mathcal{F}(\mathbb{CP}^n) \cong D^b_{sing}(\text{mirror})$ ← has $n+1$ isolated nondegenerate crit pts

$\cup$ (FOOO) $\cong$

$\left\{\begin{array}{l}\text{full subcat. gen}^{\underline{d}} \text{ by} \\ \text{Clifford torus with} \\ n+1 \text{ local systems}\end{array}\right\}$

The Clifford tori generate, by looking at $HH_* \xrightarrow{\sim} QH^*$.