M. Abouzaid – 21/1/10 – Functoriality in HMS

X smooth complex variety

D anticanonical divisor $\quad \sigma \in H^0(K_X^{-1})$, $\sigma^{-1}(0) = D$

$X \setminus D$ open CY, $\Omega = \sigma^{-1}$; also pick symplectic form on X

Strominger–Yau–Zaslow (+ Fooo, Auroux, ...)

Conj: $\exists$ special Lag. torus fibration

$$\begin{tikzcd} T^n \arrow[r, "\simeq"] & F_b \arrow[r, hook] \arrow[d] & X \arrow[d] & \supset D \arrow[d] \\ & b \in & \overline{B} & \supset \partial \overline{B} \end{tikzcd}$$

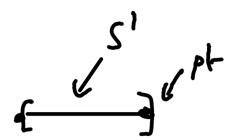
whenever b lies away from a codim. 2 subset of $\overline{B}$

$\overline{B}$ manifold with boundary, $\dim_{\mathbb{R}} B = \dim_{\mathbb{C}} X$
(corners if D normal crossings)

On D we have a T^{n-1}-fibration

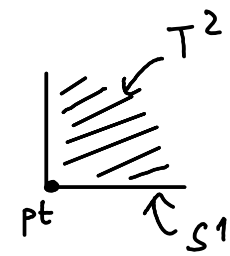
Ex: dim. 1: $\quad X = \mathbb{P}^1$
$D = \{0, \infty\}$

$$\begin{array}{ccc} z & \in & \mathbb{P}^1 \\ \downarrow & & \downarrow \\ \log|z| & \in & [-\infty, +\infty] \end{array}$$



Ex: dim. 2: locally on $\mathbb{C}^2$:
$X = \mathbb{C}^2$
$D = \{zw = 0\}$

$$\begin{array}{ccc} (z,w) & & \mathbb{C}^2 \\ \downarrow & & \downarrow \\ (|z|^2, |w|^2) & \in & [0, +\infty)^2 \end{array}$$

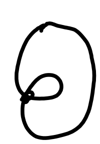


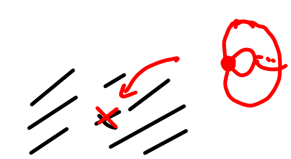
but also deformation to:

$X = \mathbb{C}^2$
$D = \{zw = \varepsilon\}$

$$\begin{array}{ccc} (z,w) & & \mathbb{C}^2 \\ \downarrow & & \downarrow \\ (|z|^2 - |w|^2,\ |zw - \varepsilon|^2) & & \mathbb{R} \times [0, \infty) \end{array}$$

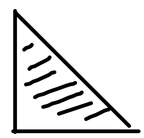
At $(0, |\varepsilon|^2)$ we have a critical value of this fibration & sing. fiber

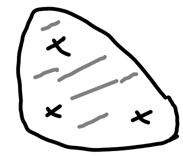




If we ignore the 'special' condition then we can patch these local models together, e.g. gluing 3 models:

($\mathbb{CP}^2$, toric divisor) deforms to ($\mathbb{CP}^2$, smooth cubic)





3 sings.

Strategy due to Gross-Siebert for producing such fibrations from toric degenerations.

If study only one half of mirr. symm., e.g. A-side, then only need Lagr. fibration $\Rightarrow$ easier to build ✓

- SYZ conj: The mirror of (X, D) is obtained by "dualizing" this fibration over $B = \overline{B} - \partial\overline{B}$:

$$\begin{array}{ccl} Y & \longleftarrow & F_b^\vee = \{U(1)\text{-loc. systems on } F_b\} = H^1(F_b, \mathbb{R}/\mathbb{Z}) \\ \downarrow & & \downarrow \\ B & \ni & b \quad \text{reg. value of fibration} \end{array}$$

together with the superpotential $W: Y \to \mathbb{C}$ which counts holom. discs with ∂ on the fibers,

$$W(b, \nabla) = \sum_u \exp\left(-\int u^*\omega + i\, hol_\nabla(\partial u)\right)$$

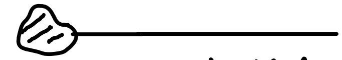
Dualization is less obvious at the singular locus.

- The problem of constructing sympl. form ω_Y from complex structure on X is local (patch together local models for (Y, ω_Y)).
- The problem of constructing complex structure on Y is NOT local. Instanton corrections — Kontsevich-Soibelman 2001
- We'll focus on: B-model on $X \longleftrightarrow$ A-model on (Y, W).

Homological mirror symm:

$\mathrm{Coh}(X) \longleftrightarrow \mathcal{F}(Y, W)$ Fukaya cat. of LG-model

objs. = Lagr. submflds of Y

st. $W(L) =$ 

compact ∪ half-line

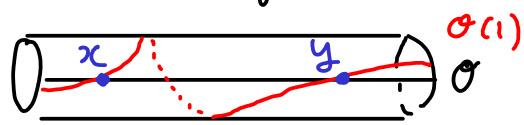
$\mathrm{Coh}(D) \longleftrightarrow \mathcal{F}(W^{-1}(p))$, $p \gg 0$

(⚠ beware if D is not smooth! then wrap fiberwise...)

$\mathrm{Coh}(X-D) \longleftrightarrow \mathcal{W}(Y)$ wrapped Fukaya category.

1) Example: $\mathbb{CP}^1 \longleftrightarrow W = z + \frac{1}{z}$ on $\mathbb{C}^*$

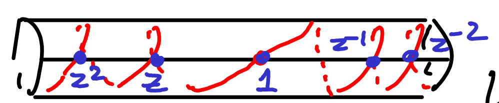
$\mathcal{O}(i) \longleftrightarrow$ isotpy class of lines from $-\infty$ to $+\infty$



$\mathcal{F}(\mathbb{C}^*, z + \frac{1}{z})$

on $\mathbb{C}^*$: sheaves $\longleftrightarrow$ isotpy classes of lines & circles,

but now all lines are isotpic!

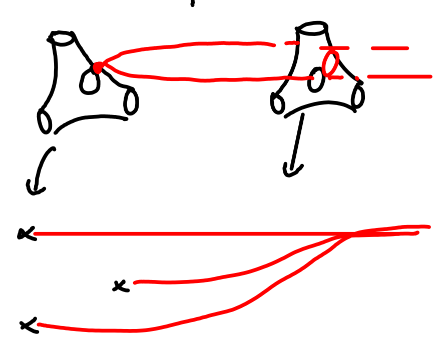
$\mathrm{End}(\mathcal{O}) = \mathbb{C}[z, z^{-1}] \longleftrightarrow$ 

$\mathcal{W}(\mathbb{C}^*)$

$HW^*(\mathcal{O}, \mathcal{O})$

2) Next: $\mathbb{CP}^2$: rel. to toric divisor $D = \{xyz = 0\}$

- $\mathbb{CP}^2 \leftrightarrow$ Mirror $(\mathbb{C}^*)^2$, $W = z + w + \frac{1}{zw}$, 3 critical points

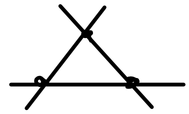


Category gen.d by thimbles:

$FS(W) \hookrightarrow \mathcal{F}(Y, W)$

(embeds; pf. of isom. not quite complete)

Seidel checked that this is mirror to an exceptional collection for $D^b\mathrm{Coh}(\mathbb{P}^2)$

- mirror to $D =$  is $\Sigma = $ 3-punctured T^2

<u>Claim:</u>
$$\mathrm{Perf}(D) \simeq \mathcal{F}(\Sigma)$$
$$\mathrm{Coh}(D) \simeq \mathcal{W}(\Sigma)$$

- $X \setminus D \simeq (\mathbb{C}^*)^2 \xleftrightarrow{\text{mirror}} (\mathbb{C}^*)^2 \cong T^* T^2$

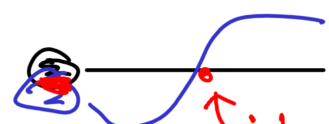
$$\mathbb{C}[z^{\pm 1}, w^{\pm 1}] \longleftrightarrow HW^*((\mathbb{R}^+)^2)$$

★ These 3 categories are related.

<u>Claim:</u> (A.–Seidel in progress) ‖ $\exists$ A_∞-functors $\mathcal{F}(W) \xrightarrow[a]{\text{acceleration}} \mathcal{W}(Y)$, $\downarrow r$ restriction, $\mathcal{F}(W^{-1}(p))$

(functor $\mathcal{F}(W) \longrightarrow \mathcal{W}(Y)$ is built by "Viterbo restriction" & viewing as a <u>quotient category</u>).

similarly, $\mathcal{F}(W) \longrightarrow \mathcal{F}(W^{-1}(p))$:



intersections for $Hom_{\mathcal{F}(W)}$: those inside $W^{-1}(p)$ form a quotient complex.

- Assume W is a <u>Lefschetz fibration</u> ie. all sings. are Morse, then $FS(W) \hookrightarrow \mathcal{F}(W)$ is a directed A_∞-category.

$$FS(W) \xrightarrow{r} \mathrm{Image}(FS(W)) =: \mathcal{B} \subset \mathcal{F}(W^{-1}(p))$$

$$r^*: \mathcal{B} \longrightarrow \mathrm{Perf}(FS(W))$$

(<u>NB:</u> Perf $\Leftrightarrow$ D^π split closure)

<u>Claim:</u> (A-S.): ‖ $\mathrm{Perf}(\mathcal{W}(Y)) \simeq \mathrm{Perf}(FS(W)/\mathcal{B})$

This is <u>mirror to</u>:

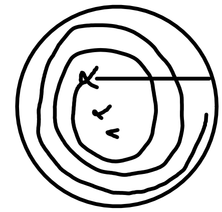
$$D^b\mathrm{Coh}(X \setminus D) = \text{localization of } D^b\mathrm{Coh}(X) \text{ at } D^b\mathrm{Coh}_D(X)$$

(supported at D)

<u>Strategy:</u> 0) prove that thimbles split-generate $\mathcal{W}(Y)$ (tomorrow)

1) compare morphisms in $\mathcal{W}(Y)$ & quotient.

<u>Point:</u> morphisms in $\mathcal{W}(Y)$ = bend thimbles around ∞ many times



> geometrically, the Serre functor in $FS(W)$ corresponds to "bending" once; moreover, $\exists$ natural transf$^{\underline{n}}$ Serre $\to$ id defined by counting sections of W.

[Seidel]