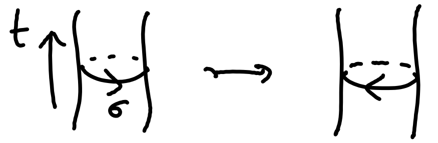
K. Hori – 18/1/10 – Mirror symmetry and reality

- (M,τ) space with an involution

⇝ "orientifold" theory obtained by gauging σ-model

$$(X:\Sigma\to M)\ \longmapsto\ (\tau\circ X\circ\Omega:\Sigma\to M)$$

where $\Omega:\Sigma\to\Sigma$ orient.-reversing involution.



in closed string theory

$$(t,\sigma)\mapsto(t,-\sigma)$$

In top string theory, when τ antiholomorphic, this relates to eg. Welschinger invariants, Solomon, ...

- An <u>orientifold plane</u> is a conn. component Y of M^τ, together with

N D-branes at Y, ie. $O(N)$ or $USp(N)=Sp(N/2)$

(say O-plane is of type O^+ or O^- depending which).

Orientifolds in Type II string theory:

M 10-dim'l Riem. mfld of signature 9+1, with spin structure.
(real and/or complex)

τ involution lifts to real spinor bundle as $\tilde\tau: S(M)\to S(M)$
(assume τ preserves time direction)

τ can be orient. preserving: $\tilde\tau^2=\begin{cases}+id & (B_+)\\ -id & (B_-)\end{cases}$ } IIB strings

or orient. reversing: $\tilde\tau^2=\begin{cases}+id & (A_+)\\ -id & (A_-)\end{cases}$ } IIA.

Possible O-planes: (codim. k ⇝ $O(9-k)$-plane)

(B_+)	codim $=k=0$ mod 4	⇒ O9/O5/O1 planes
(B_-)	2	O7/O3
(A_+)	3	O6/O2
(A_-)	1	O8/O4/O0.

<u>Data:</u> | E herm. vect bundle/M, A unitary conn., $T\in End(E)$ Hermitian endo.
IIA: E ungraded, IIB: $E=E^0\oplus E^1$ $\mathbb{Z}/2$-gr., $A=\begin{pmatrix}A^0 & \\ & A^1\end{pmatrix}$, $T=\begin{pmatrix}0 & *\\ * & 0\end{pmatrix}$

Orientifold action on (E, A, T):

$$A \mapsto -\tau^*A^t + \alpha\, \mathrm{id}_E \qquad \text{on } E \mapsto \tau^* \overset{\text{dual}}{E^*} \otimes \overset{\text{line bundle}}{\mathcal{L}}$$
$$T \mapsto \pm \tau^* T^t$$

where α is constrained by $d\alpha = B + \tau^* B$ where B = B-field

"worldsheet action":

$$S_B = \int_\Sigma X^*B \mapsto \int_\Sigma (\tau \circ X \circ \Omega)^* B = -\int_\Sigma X^* \tau^* B$$

hence $\Delta S_B = -\int_\Sigma X^*(B + \tau^* B)$

- If $\partial\Sigma = \emptyset$ (closed string): need $e^{i\Delta S_B} = 1$

$$\Leftrightarrow [\tau^*B + B] \in H^2(X, 2\pi\mathbb{Z})$$

ie. $[\tau^*B + B] = -2\pi c_1(\mathcal{L})$ for some line bundle $\mathcal{L} \to M$

ie. $\exists$ $U(1)$-conn. α st. $\tau^*B + B = d\alpha$ (see above).

- If $\partial\Sigma \neq \emptyset$ (open string):

$$\Delta S_B = -\int_\Sigma X^*(B + \tau^*B) = -\int_\Sigma X^* d\alpha = -\int_{\partial\Sigma} X^*\alpha$$

This leads to the shift of A by α so action preserved.

(recall A enters into e^{iS} as the parallel transport $\mathcal{P}\exp\left(-\int_{\partial\Sigma} X^*A\right)$).

Orientifold action on open string states:

An open string state Φ is an assignment

$$X \in \mathrm{Map}([0,1], M) \mapsto \Phi[X] \in \mathrm{Hom}_{\mathbb{C}}(E_{X(0)}, E_{X(1)}).$$

Natural action of parity: $\Phi[X] \longmapsto \mathcal{P}(\Phi)[X] := \Phi[\tau . X . \Omega]^t$

$$\in \mathrm{Hom}\Big(\underset{\tau^*E^*_{X(0)}}{\underset{\parallel}{E^*_{\tau X \Omega(1)}}}, \underset{\tau^*E^*_{X(1)}}{\underset{\parallel}{E^*_{\tau X \Omega(0)}}}\Big)$$

where reversal $\Omega : [0,1] \to [0,1]$
$\sigma \mapsto 1 - \sigma$.

However want $\tau^*E^* \otimes \mathcal{L}$, not τ^*E^* $\Rightarrow$ modify to:

$$\mathcal{P}(\Phi)[X] := \Phi[\tau X \Omega]^t \otimes \mathcal{P}\exp\left(-\int_0^1 X^*\alpha\right) \in \mathrm{Hom}(\tau^*E^*_. \otimes \mathcal{L}_{X(0)}, \tau^*E^* \otimes \mathcal{L}_{X(1)})$$

Still not quite what we want, as we'd like $\Phi \mapsto P[\phi]$ to select invariant string states, hence need to compare $\phi(X)$ & $P(\phi)[X]$

Need an isomorphism $U: (\tau^* E^* \otimes \mathcal{L}, -\tau^* A^t + \alpha, \pm \tau^* T) \xrightarrow{\sim} (E, A, T)$.

Then set $P(\phi)[X] = U(X(1)) \circ \mathcal{P}(\phi)[X] \circ U(X(0))^{-1} \in \mathrm{Hom}_{\mathbb{C}}(E_{X(0)}, E_{X(1)})$

Need $P^2 = id$; calculate

$$P^2(\phi)[X] = U(\tau^* U^t)^{-1}_{|X(1)} \circ \phi[X] \circ U(\tau^* U^t)^{-1}_{|X(0)}$$
$$\otimes \exp\left(-\int_0^1 X^*(\alpha - \tau^*\alpha)\right).$$

$\Rightarrow$ want: $U(\tau^* U^t)^{-1} = c \cdot id$, where c is a section of $\tau^* \mathcal{L} \otimes \mathcal{L}^{-1}$

Need: c is a global parallel section wrt $\tau^*\alpha - \alpha$.

Hence: at an O-plane Y, $\mathcal{L}_{|Y} \underset{\text{canonically}}{\cong} \tau^* \mathcal{L}_{|Y}$; we want

$c: \mathcal{L} \cong \tau^* \mathcal{L}$ over M, with the two isomorphisms equal over Y up to a sign ("O-plane type").

gauge transformations act by

$$(B, \mathcal{L}, \alpha, c) \mapsto (B + d\Lambda, \mathcal{L} \otimes \Lambda, \alpha + \Lambda + \tau^*\Lambda, c)$$

$$(E, A, T, U) \mapsto (E \otimes L, A + \Lambda, T, U\Lambda).$$

Mirror symmetry: $M = \underset{\uparrow\ \text{Kähler mfld}}{X^{2n}} \times \mathbb{R}^{(9-2n)+1}$

σ-model on X has $N=(2,2)$ supersymmetry

$\tau: X \to X$ preserves a half of it:

(B) want τ holomorphic

(A) τ antisymplectic

Mirror symmetry : $\underset{\text{holom.}}{(X, \tau, B, \mathcal{L}, \alpha, c)} \longleftrightarrow \underset{\text{antisympl.}}{(X', \tau', B', \mathcal{L}', \alpha', c')}$

<u>Ex:</u> $X = T^2 = \mathbb{R}^2/\mathbb{Z}^2$

involution	type	fixed pts	B-field	$(\mathcal{L}, \alpha)$:	O-plane:
$(x,y)\mapsto(x,y)$	B	T^2	$\nearrow 0 \longrightarrow$	trivial $\longrightarrow$	$\nearrow O9^-$ $O(N)$ $\searrow O9^+$ $USp(N)$
			$\searrow \pi\, dx\wedge dy \longrightarrow$	$\mathcal{O}(-p)$ any p $\longrightarrow$	$\nearrow O9^-$ $O(N)/\pm1$ $\searrow O9^+$ $USp(N)/\pm1$
$(x+\frac{1}{2},y)$	B	$\emptyset$	0	triv	$\emptyset$
$(-x,y)$	A	$\{0,\frac{1}{2}\}\times S^1$	arbitrary	$\nearrow$ triv $\longrightarrow$	$\nearrow O8^-\times2$ $\searrow O8^+\times2$
				$\searrow$ 2-torsion $\longrightarrow$	$O8^+ + O8^-$
$(-x,y+\frac{1}{2})$	A	$\emptyset$	arbitrary	triv	$\emptyset$
(y,x)	A	$S^1_{\text{diag.}}$	arbitrary	triv	$\longrightarrow O8^-, O8^+$
$(-x,-y)$	B	4 pts $\{0,\frac{1}{2}\}^2$	$\nearrow 0$	$\nearrow$ triv	$\longrightarrow O7^-\times4, O7^+\times4$
				$\searrow$ 2 torsion	$\longrightarrow O7^-\times2, O7^+\times2$
			$\searrow \pi\, dx\wedge dy$	$\mathcal{O}(-p)$, $\tau(p)=p$	$\longrightarrow O7^+_-$ at p, $O7^-_+\times3$.

Mirror symmetry interchanges these cases explicitly. ⟷ ⟷ ⟷

(several possibilities since can do T-duality in either direction x or y).