K. Fukaya – 22/1/10 – Cyclic symmetry and numerical invariants in Lagr. Floer theory, part II*: CY case

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(cf. 0907.4219, 0908.0148) (* part I: Fano case was in Korea last week)

- C finite dim$^{\underline{l}}$ vector space, or $C = \Omega^\bullet(L)$ de Rham complex, $\dim L = n$
  work over $\Lambda = \{\sum a_i T^{\lambda_i},\ a_i \in \mathbb{R}\}$
- $\langle u,v\rangle : C^k \otimes C^{n-k} \to \Lambda$ st. $\langle u,v\rangle = (-1)^{1+\deg' u \deg' v}\langle v,u\rangle$
  where $\deg' = \deg + 1$.

  (in dR case, $\langle u,v\rangle = (-1)^{\deg u.(\deg v+1)} \int_L u \wedge v$)

- Filtered cyclic A_∞-alg: $(C, \{m_k\}_{k=0}^{\infty}, \langle\cdot,\cdot\rangle)$ where $m_k :=$

$$m_{k,\beta} : C[1]^{\otimes k} \longrightarrow C[1] \quad \text{of degree} = 1 - \mu(\beta)$$

where $\mu : G \to 2\mathbb{Z}$ (Maslov index)
$E : G \to \mathbb{R}_{\geq 0}$ proper map (energy)

$$m_k = \sum_\beta T^{E(\beta)} m_{k,\beta}$$

satisfying
$$\begin{cases} 1) \displaystyle\sum_{\substack{k_1+k_2=k \\ i}} \pm m_{k_1}(x_1 \dots, m_{k_2}(x_i, \dots), \dots) = 0 \\ 2) \langle m_k(x_1 \dots x_k), x_0\rangle = (-1)^* \langle m_k(x_0 \dots x_{k-1}), x_k\rangle \end{cases}$$

Thm: (FO^3 + ...)

$L \subset M$ Lagr. submfld, rel. spin $\Rightarrow (\Omega^\bullet L \otimes \Lambda, \langle\cdot,\cdot\rangle)$ can be equipped with $\{m_k\}$ giving it a structure of cyclic filtered A_∞-alg, well-def^{d} up to pseudo-isotopy.

- Homotopy equiv^{ce}:

$$f : (C, m, \langle\,\rangle) \longrightarrow (C', m', \langle\,\rangle') = \{f_k : C[1]^{\otimes k} \to C'[1]\}$$

an A_∞-quasi-isomorphism, compat. with $\langle\cdot,\cdot\rangle$, ie:

s.t. 1) $\sum m'_\ell(f_{k_1}(\cdots),\dots,f_{k_\ell}(\cdots)) = \sum f_k(\cdots, m_\ell(\cdots),\cdots)$

(A_∞-homomorphism)

2) $\displaystyle\sum_{k_1+k_2=k} \langle f_{k_1}(x_1\cdots), f_{k_2}(\cdots x_k)\rangle = \begin{cases} 0 & \text{for } k\neq 2 \\ \langle x_1, x_2\rangle & \text{for } k=2. \end{cases}$

3) quasiisom., i.e. isom on H^*.

- Pseudo-isotopy :=

<u>Def.</u> $(C,\{m_k^t\},\{c_k^t\},\langle\cdot,\cdot\rangle)$ is a <u>pseudoisotopy</u> if:

consider $C\hat{\otimes}\Omega^*([0,1]) \ni x_i = a_i(t) + b_i(t)\,dt$;

equip it with $m_k(x_1\dots x_k) = x + y\,dt$, where

$x(t) = m_k^t(a_1(t)\dots a_k(t))$

$$y(t) = \begin{cases} \sum_i m_k^t(a_1\dots b_i(t)\dots a_k) + c_k^t(a_1(t)\dots a_k(t)) & k\neq 1 \\ m_1^t(b_1(t)) + c_1^t(a_1(t)) + \dfrac{da_1}{dt} & k=1. \end{cases}$$

Require: • $(C\hat{\otimes}\Omega^\circ([0,1]), m_k)$ A_∞-relations

• m_k^t, c_k^t cyclically symmetric.

Say $(C,\{m^0\},\langle,\rangle) \underset{\text{pseudo iso}}{\sim} (C,\{m^1\},\langle,\rangle)$ if $\exists$ pseudoisotopy with $m^t|_{t=0} = m^0$, $m^t|_{t=1} = m^1$.

<u>Rmk.</u> • Pseudoisotopic $\Rightarrow$ homotopy equivalent but converse is presumably false.

• Continuation in Floer theory gives pseudoisotopy.

Now focus on CY3 case, i.e. $n=3$, and $\mu: G\to\mathbb{Z}$ is $\mu\equiv 0$.

Hence $\deg m_{k,\beta} = 1$ $\forall\beta$ after shift

<u>Superpotential:</u> $\Psi': C^1 \to \Lambda_0$, $\displaystyle \Psi'(b) = \sum_{k=0}^\infty \frac{1}{k+1}\langle m_k(b\dots b), b\rangle$

Maurer-Cartan scheme: $\widetilde{\mathcal{M}}(C) = \{ b \in C^1 / \sum_{k=0}^{\infty} m_k(b \ldots b) = 0 \}$

(equivalently: deformed $m_0^b = 0$).

$\sim$ gauge equivalence; $\mathcal{M}(C) = \widetilde{\mathcal{M}}(C)/\sim \; \subset H^1(C, m_0^1) = H^1(L; \Lambda_0)$

Lemma: $b \in \widetilde{\mathcal{M}}(C) \iff b$ is a critical point of ψ'.

(easy, using cyclic symmetry to calculate $D\psi'$).

Problem: $(C, \{m_k^t\}, \{c_k^t\}, \langle \cdot, \cdot \rangle)$ pseudo isotopy

$f^t : (C, \{m_k^0\}) \to (C, \{m^t\})$ induced homotopy equivalences

Then $f_*^t : \mathcal{M}(C, m^0) \xrightarrow{\sim} \mathcal{M}(C, m^t)$.

Question: $\Psi_t'(f^t(b)) \overset{?}{=} \Psi_0'(b)$? ANSWER: NO

Compute: Let $b_t = f_*^t(b) := \sum_k f_k^t(b \ldots b)$

$$\frac{d}{dt} \Psi_t'(b_t) = \frac{d}{dt} \sum_k \frac{1}{k+1} \langle m_k^t(b_t \cdots b_t), b_t \rangle$$

$$= \sum_{k_1 + k_2 \geq 1} \langle c_{k_1}^t(b_t \cdots b_t), m_{k_2}^t(b_t \cdots b_t) \rangle$$

$$= - \langle c_0^t(1), m_0^t(1) \rangle$$

since, if sum included $k_1 = k_2 = 0$, the sum would be $\langle \sum c_{k_1}^t(b_t \ldots b_t), \overbrace{\sum m_{k_2}^t(b_t \cdots b_t)}^{=0} \rangle$

Def: • $(C, \{m_k\}, \langle \, \rangle, m_{-1})$ is an inhomogeneous cyclic filtered A_∞-alg.

if: $\begin{cases} \bullet \ (C, \{m_k\}, \langle \, \rangle) \text{ cyclic filtered } A_\infty\text{-alg.} \\ \bullet \ m_{-1} \in \Lambda_+ \end{cases}$

• $(C, \{m_k^t\}, \{c_k^t\}, \langle \, \rangle, m_{-1}^t)$ is a pseudoisotopy if

• $(C, \{m_k^t\} \{c_k^t\}, \langle \cdot, \cdot \rangle)$ pseudo-iso of cyclic filtered A_∞-algs.

• $dm_{-1}^t / dt = \langle c_0^t(1), m_0^t(1) \rangle$

Now define $\Psi : \mathcal{M}(C, \{m_k\}) \longrightarrow \Lambda_0$

$$\Psi(b) = \sum_k \frac{1}{k+1} \langle m_k(b \ldots b), b \rangle + m_{-1}.$$

Lemma: $(C, m^0, \langle \cdot, \cdot \rangle, m^0_{-1}) \underset{p\text{-iso.}}{\sim} (C, m^1, \langle \cdot, \cdot \rangle, m^1_{-1})$

$\Rightarrow \Psi^1(f_*(b)) = \Psi^0(b)$ on $\mathcal{M}(C, \{m^0\})$

[NB: idea of m_{-1} comes from D. Joyce]

Thm: $L \subset M$, $c_1(M) = 0$, $\dim_{\mathbb{C}} M = 3$; L Lagr., rel. spin, $\mu_L = 0$

J almost. cx str. s/t suitable condition $(*)$

$\Rightarrow$ can define $(\Omega(L), \{m_k\}, \langle \cdot, \cdot \rangle, m_{-1})$ inhom. cyclic filtered A_∞-alg., well-def. up to pseudo isotopy (still fixing J)

Condition $(*)$:

$$\begin{array}{c} \mathcal{M}_1(\alpha, J) = \left\{ u : S^2 \to M \,\middle/\, \begin{array}{l} J\text{-holom.} \\ [u] = \alpha \end{array} \right\} / Aut(S^2, z_0) \\ \big\downarrow {\scriptstyle ev : u \mapsto u(z_0)} \\ M \end{array}$$

Require $ev(\mathcal{M}_1(\alpha, J)) \cap L = \emptyset \quad \forall \alpha \neq 0.$

For fixed J, $\Psi_J : \mathcal{M}(C, m) \longrightarrow \Lambda_0$ is an invariant

Remains constant if vary J while keeping $(*)$.

In general, jumps by wall-crossing!

J_0, J_1 acs satisfying $(*)$, $\mathcal{J} = [J_t]_{t \in [0,1]}$

$$\begin{array}{c} \mathcal{M}_1(\alpha, \mathcal{J}) = \bigcup_t \mathcal{M}_1(\alpha, J_t) \\ \big\downarrow {\scriptstyle ev} \\ M \end{array}$$

$n(\alpha) := \#(L \cap ev_* \mathcal{M}_1(\alpha, \mathcal{J})) \in \mathbb{Q}.$

Then consider:

$$\begin{tikzcd} b \in \mathcal{M}(L,J_0) \arrow[r, "f_*", "\sim"'] \arrow[d, "\Psi_{J_0}"] & \mathcal{M}(L,J_1) \arrow[d, "\Psi_{J_1}"] \\ \Lambda_0 & \Lambda_0 \end{tikzcd}$$

<u>Thm:</u> (wall-crossing formula):

$$\Psi_{J_0}(b) - \Psi_{J_1}(f_*(b)) = \sum_\alpha n(\alpha)\, T^{E(\alpha)}$$

<u>Relation to earlier works:</u>

1) <u>Solomon</u>'s thesis (cf. Welschinger):

$\dim M = 3$, $c_1 M = 0$, $\tau : M \to M$, $\tau^2 = 1$, $\tau^* J = -J$

$L = \{x \in M \,/\, \tau(x) = x\}$

$\forall \beta \in \pi_2(M,L)$, $\mathcal{M}(\beta) = \{u : (D^2,\partial) \to (M,L) \ \substack{J\text{-hol.} \\ [u]=\beta} \}/\mathrm{Aut}\, D^2$

$$n_\beta = \# \mathcal{M}(\beta)$$

$\rightsquigarrow$ Solomon shows $\underline{\sum n_\beta T^{E(\beta)}}$ is well-def^d invariant.

<u>FOOO:</u> $\tau : M \to M$ antiholom. involution, $L = \mathrm{Fix}\, \tau$

$\Rightarrow m_k : \Omega(L)^{\otimes k} \to \Omega(L)$ satisfies

$$m_{k,\beta}(x_1 \dots x_k) = (-1)^* m_{k,\tau(\beta)}(x_k \dots x_1) \quad \forall \beta \in \pi_2(X,L)$$

where $* = \dfrac{\mu(\beta)}{2} + k + 1 + \sum_{i<j} \deg' x_i \deg' x_j$

(look at how τ acts $\mathcal{M}_k(\beta; x_1 \dots x_k) \xrightarrow{\sim} \mathcal{M}_k(\tau(\beta); x_k \dots x_1)$)
& orientation issues ...

For $\mu_L = 0$ and $k = 0$, get: $m_0(1) = -m_0(1) = 0$.

Hence 0 is a Maurer-Cartan element.

<u>Conj:</u> Solomon's inv^t $= \Psi_J(0)$.

NB: independence of J is ok, due to cancellations in wall-crossing formula b/w n_α and $n_{\tau(\alpha)}$

Lemma: J_0, J_1 τ-anti-invariant, isotopic through anti-inv't J_t
$\Rightarrow \Psi_{J_0}(0) = \Psi_{J_1}(0)$.

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- $L \subset M$, S^1 acts on M, freely on L
  $\beta \in \pi_2(M,L)$, $\dim \mathcal{M}(\beta) = 0$
  $\Rightarrow \exists$ well-defined $\# \mathcal{M}(\beta) \in \mathbb{Q}$ (indep't of J, but may depend on the S^1-action)

  (uses S^1-equivariant perturbation & fact that S^1 acts freely on $\partial \mathcal{M}(\beta)$)

- If $L \subset M$, S^1 acts freely on L, assume $c_1(M) = 0$, dim. 3 and $\mu_L = 0$ on $\pi_2(M,L)$
  ($\Rightarrow \dim \mathcal{M}(\beta) = 0 \ \forall \beta$)
  $[\gamma] :=$ class of an S^1-orbit $\in H_1(L, \mathbb{Z})$

  NB: $\# \mathcal{M}(\beta) \neq 0 \Rightarrow [\partial \beta] = k[\gamma]$ for some k.

$x = [\gamma] \cdot - : H^1(L, \Lambda_0) \longrightarrow \Lambda_0$

$y = e^x : H^1(L, \Lambda_0) \to \Lambda_0$ (i.e. compose with $\exp : \Lambda_0 \to \Lambda_0$)

$$\Phi = \sum_k \sum_{\partial\beta = k[\gamma]} \# \mathcal{M}'(\beta)\, y^k\, T^{E(\beta)} : H^1(L, \Lambda_0) \to \Lambda_0$$

Conj: 1) $\{ b \,/\, \nabla \Phi(b) = 0 \} = \mathcal{M}(L)$
2) $\Psi_L = \Phi$ on $\mathcal{M}(L)$ for $J = S^1$-inv't cx. structure

(2) says: Counting discs with coeff't $\exp(k[\gamma] \cdot b) \Longleftrightarrow \sum m_\ell(b \ldots b)$

(Rmk: some indep't work by Iacovino)