G. Mikhalkin – 23/1/10 – (p,q) homology classes in tropical geometry

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**Goal:** $X$ tropical mfld (w/ Itenberg + Zharkov) (cf. also Katzarkov)

$\rightsquigarrow$ bigraded homology/cohomology

st. $\dim H_{p,q} = h^{p,q}$ (generic fiber of complex 1-param. family approximating $X$)

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Tropical geom:

E.g.: suppose we have a family $\mathcal{X}_t$, $t \in \Delta^*$ with a tropical limit

(metric graph)

trop. limit

(depends on parametrization $t$)

Toric geometry: $\mathbb{CP}^n \rightsquigarrow \mathbb{TP}^n$ = simplex, glued from $\mathbb{T}^n$

$\log_t$

moment map

$\mathbb{T} = [-\infty, \infty)$

"$x+y$" $= \max(x,y)$

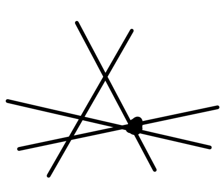
"$xy$" $= x+y$

* tropical Laurent polynomials wrt "+", "·" define $\mathcal{O}$ (sheaf of tropical regular functions).
  (can be geometrically realized as $\mathbb{Z}$-affine structure on top dim. strata)

* Tropical manifold: $X^n$ = union of convex $n$-polyhedra (with $\mathbb{Z}$ structure) (polyhedral complex)

$X$ is a smooth trop. mfld in a coarse sense if at each point it is given by a matroid ($\Leftrightarrow$ comes from a simply balanced embedding to $\mathbb{T}^n$).

Matroids = (virtual) hyperplane arrangements in $\mathbb{P}^n$



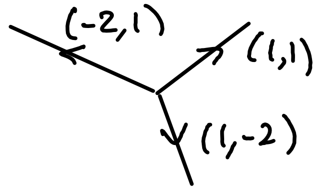
(also make some combinatorial arrangements that are not realizable)

By work of Bergman, Ardila–Klivans, ...

matroids ⇒ possible local models for junction pts in a tropical manifold.

(Also ⟺ <u>simply balanced</u> := balanced + ∃ linear subspace of complementary dimension whose local intersection number is 1.

<u>Ex:</u>

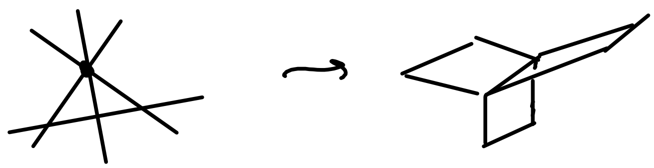



is balanced but not simply bal.)

<u>NB:</u> generic arrangement =

"generic trop. vertex"

but



(note: ∃ $\mathbb{R}$-factor in complement).

<u>idea:</u> $\mathbb{P}^n \setminus$ (hyperplanes) $\underset{h.e.}{\simeq}$ complex.

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- Given a trop. mfld $X$,

  $\mathcal{F}_j \subset \Lambda^j(\mathbb{R}^n)$ generated by $j$-dim'l cones in $X$ defines a local system over $X$

  We also have the dual local system $\mathcal{F}^\bullet$ ($\rightsquigarrow$ cohomology)

  $\mathcal{F}_\bullet$ is good for building $H_\bullet(X)$ in the sense that, given two strata $\underset{\text{polytope face}}{\Delta \supset \Delta'}$, we have a projection $\mathcal{F}_\bullet(\Delta) \to \mathcal{F}_\bullet(\Delta')$

  $\mathcal{F}^\bullet(\Delta) \leftarrow \mathcal{F}^\bullet(\Delta')$

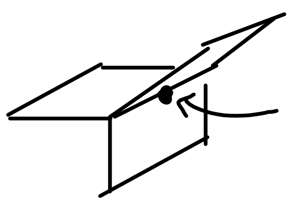
  so $\mathcal{F}_\bullet$ cosheaf, $\mathcal{F}^\bullet$ sheaf

(These maps go in that direction because
$\mathcal{F}_\bullet(\Delta)$ = look at open star of $\Delta$
$\Delta \supset \Delta' \Rightarrow$ star of $\Delta \subset$ star of $\Delta'$).
$\Rightarrow$ homology needs $\mathcal{F}_\bullet(\Delta) \to \mathcal{F}_\bullet(\Delta')$).

<u>Def:</u>
$$H_{p,q}(X) = H_q(X, \mathcal{F}_p)$$
$$H^{p,q}(X) = H^q(X, \mathcal{F}^p)$$

- $\exists$ other model: $W_p(pt) = \Lambda^\alpha$(maximal linear space locally contained inside $X$).

Now for $\Delta \supset \Delta'$, get $W_\bullet(\Delta') \to W_\bullet(\Delta)$
$W^\bullet(\Delta) \to W^\bullet(\Delta')$

<u>Ex:</u>  $\mathcal{F}_1 = \mathbb{C}^3$
$W_1 = \mathbb{C}$

- We have a pairing $W_j \times \mathcal{F}_k \to \mathcal{F}_{j+k}$.

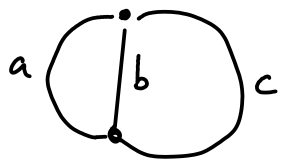
<u>Conjecture</u> (Thm in progress)

$H_q(X, \mathcal{F}_p) \simeq$ weight filtration given by Schmid thm on $\mathcal{X}_t$
where $\mathcal{X}_t$ = family over punctured disk cv to $X$.

ie. each $H_q(X, \mathcal{F}_p)$ is $\simeq H_{p,q}(\mathcal{X}_t)$, and moreover the Gauss-Manin monodromy operator on weight filtration $W_j/W_{j-1} \xrightarrow{\phi} W_{j-1}/W_{j-2}$ ($\phi$ = monodromy $-$ Id) can be realized by a "<u>tropical wave</u>".

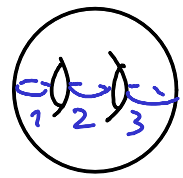
<u>Tropical wave</u> $:= \Psi \in H^1(X, W_1) = H^1_1$
which, using the above, acts by $H_{p,q} \times H^1_1 \to H_{p+1,q-1}$

Ex:



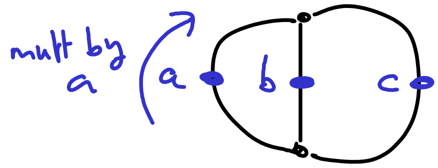

trop. genus 2 curve,
lengths of edges $= a, b, c \in \mathbb{Z}_+$

= limit of a family




with monodromy $= \tau_1^a \tau_2^b \tau_3^c$

Wave is given by




i.e. elt of $H^1(W_1)$ which has gluing = mult. by $\begin{matrix} a \\ b \\ c \end{matrix}$ through 1st blue point

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Understanding $H_{p,q}$:

- easiest part: $\mathcal{F}_0 = W_0 = \mathbb{Z}$ : so $H_{0,q} = H_q(X)$

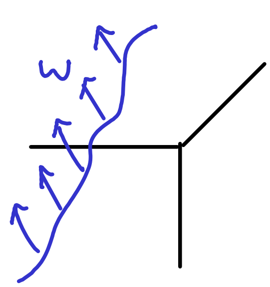
  in particular $H_{0,n} = H_n(X) = \mathbb{Z}$ for CY ✓

- $H_{1,1}$ of quadric surface:

  quadric = 2 floors connected by a conic ...

  .....

- Think of wave $Q \in H_1^1 = H^1(X, W_1)$




as a something that can be evaluated on a path $\Gamma$ together with a framing $W$

$$Q(\Gamma) := \int_\Gamma W$$