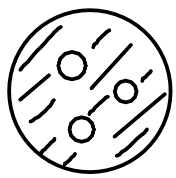
D. Kaledin – 18/1/10 – 2-algebras as factorization algebras

Deligne conjecture concerning the algebraic structure on Hochschild complex $CH^\bullet(A)$, when A assoc. algebra/k (char $k = 0$). ?

($HH^\bullet(A)$ is a Gerstenhaber algebra, but want chain level structure).

Answer in terms of operad of small discs



n smaller discs inside D^2 $\rightsquigarrow$ configuration space O_n (operad of top. spaces).

$H_\bullet(O_n, k)$ is the Gerstenhaber operad.

$C_\bullet(O_n, k)$ dg-operad, algebras over this operad are "2-algebras".
(however this is too large !).

Thm: $CH^\bullet(A)$ is a 2-algebra "up to a quasi isom"

Factorization algebras:

D unit disc, S finite set; $D^S = \mathrm{Maps}(S, D)$

- $\forall$ map $S \to D$, $\exists!$
$$\begin{tikzcd} S \arrow[rr] \arrow[dr, two heads, "f"'] & & D \\ & S' \arrow[ur, hook] & \end{tikzcd}$$
; D^S is stratified by f's (ie., by which pts of S are mapped to same pt of D).

$D^S_0 \subset D^S$ open stratum
(ie. injective maps $S \to D$)

- $f : S \twoheadrightarrow S'$ induces $i_f : D^{S'} \to D^S$ inclusion

$D^S_f \overset{j_f}{\hookrightarrow} D^S$ open subset ($\supset D^S_0$) is the union of all $i_g(D_0^{S_1}) \subset D^S$, for
$$\begin{tikzcd} S \arrow[r, "g"] \arrow[rr, bend right, "f"'] & S_1 \arrow[r] & S' \end{tikzcd}$$

(ie. maps $S \to D$ st equalities can only occur within partition classes given by fibers of f:
$D^S_{\text{constant map}} = D^S$, $D^S_{id : S \to S} = D^S_0$)

Def: A <u>factorization algebra</u> is a collection of the following.

(1) $\forall S$, $\mathcal{E}_\bullet(S)$ complex of constr.-sheaves of k-vect spaces on D^S, locally constant over strata.

(2) $i_f^! \, \mathcal{E}_\bullet(S) \xrightarrow{\sim} \mathcal{E}_\bullet(S')$ quasi isom. $\forall f : S \to S'$

(3) $j_f^* \, \mathcal{E}_\bullet(S) \xrightarrow{\sim} \boxtimes_{s \in S'} \mathcal{E}_\bullet(f^{-1}(s))$

<u>NB:</u> $S = \coprod_{s \in S'} f^{-1}(s)$, so $D^S = \prod_{s \in S'} D^{f^{-1}(s)}$

<u>Ex:</u> • for $S = \{pt\}$, $A_\bullet = \mathcal{E}_\bullet(pt)$

• for $|S| = 2$, $D^S = D^2$, $D \subset D^2$ diagonal
$U = D^2 \setminus D$ open stratum

need $i_f^! \, \mathcal{E}_\bullet(S) \cong \mathcal{E}_\bullet(pt)$ (on the diagonal)

and $\mathcal{E}_\bullet(S)_{|U} \cong \mathcal{E}_\bullet(pt)^{\boxtimes 2} = A_\bullet^{\boxtimes 2}$ (away from diagonal)

gluing these two $\longleftrightarrow$ a map $i_f^! \, j_{f!} \left(\mathcal{E}_\bullet(pt)^{\boxtimes 2} \right) \longrightarrow \mathcal{E}_\bullet(pt)$

$$i_f^! \, j_{f!} A^{\boxtimes 2} \cong \begin{array}{c} C_\bullet(U, A_\bullet^{\boxtimes 2}) \\ \downarrow \\ A_\bullet \end{array} = \left(H_*(S^1, k) \otimes A^{\otimes 2} \right)_{\mathbb{Z}/2\mathbb{Z}}$$

using $U \underset{h.e.}{\sim} S^1$.

Compare with 2-algebras.

<u>NB:</u> D^S_0 is homotopy equiv to O_S from small disc operad (shrink discs to pts).

$\rightsquigarrow$ <u>Conjecture:</u> The category of factoriz^n algebras "up to quasi iso" is equivalent to the category of 2-algebras up to q.iso.

(issues seem purely technical)

Observation: Factorization algebras are the same as chiral algebras except that (1) replace constructible sheaves w/ D-modules
(2) replace complexes w/ objects
quasi iso $\quad$ isom.

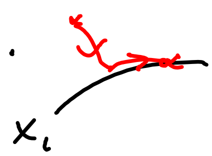
Combinatorial models:

1. X stratified space, nice enough, st. open strata are $K(\pi,1)$'s.

<u>Stratified fund! category</u> $\overline{\pi_1(X)}$: (not a groupoid)

objects $x \in X$

morphisms: $\gamma : [0,1] \to X$ st. $\forall X_1 \subset X$ (closure of a stratum),
$\gamma^{-1}(X_1) = [a,1]$ for some $0 \le a \le 1$.

(ie.  once enter a stratum, remain stuck in it).

Then $D(\mathrm{Shv}_{\mathrm{constr}}(X)) \simeq D(\mathrm{Fun}(\overline{\pi_1(X)}, k\text{-Vect}))$.

$$\left\{ \begin{array}{l} x \in X \longmapsto E_x \in k\text{-vect} \\ x \overset{\gamma}{\rightsquigarrow} y \longmapsto \text{map } E_x \to E_y \\ y \text{ in lower stratum} \end{array} \right\}$$

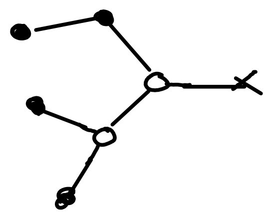
For D^S, $\overline{\pi_1(D^S)}$ is easy to describe.

$$\left\{ \begin{array}{l} f : S \twoheadrightarrow S' \text{ correspond to isom. classes of objects} \\ \mathrm{Aut}(\mathrm{Id}) = B_S \text{ pure braid group} \\ \mathrm{Aut}(f) = \prod_{s \in S'} B_{f^{-1}(s)} \end{array} \right.$$

In the definition of fact! algebra, one can replace D^S with $\overline{\pi_1(D^S)}$ everywhere and get a purely combinatorial def$^{\underline{n}}$.

(cf. Segal description of loop spaces:
1) $O_n^{E^\infty} \times X^n \to X$, $\quad$ 2) $\begin{array}{ccc} X_n & \longrightarrow & X \\ \downarrow \wr & & \\ X^n & & \end{array}$)

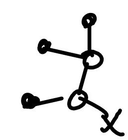
2. $\exists$ another combinatorial model using <u>planar trees</u>. ("T^S")



- Consider planar trees with a single root, marked by S ie. $S \hookrightarrow \mathrm{Vertices}(T)$
- with stability condition: unmarked vertices have valency ≥ 3

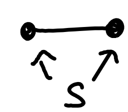
D^S_0 has a stratification with strata $\longleftrightarrow$ S-marked stable trees.

<u>Idea</u>: think of (D,S) as Riem. surf., consider a holom. quadratic differential & look at trajectories of induced sing. foliations.

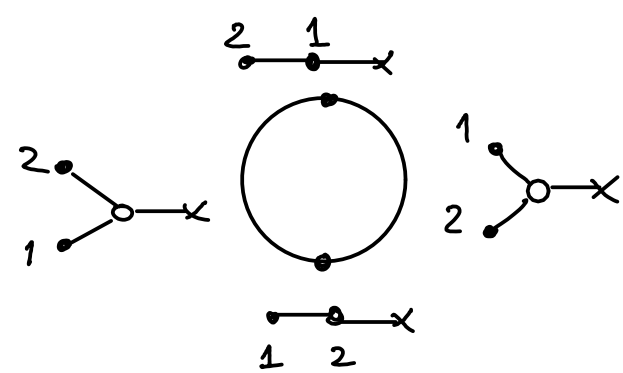
<u>Open strata</u> = trivalent trees. with only the leaves marked 

<u>Adjacency order</u>: can contract edges w/ ≥ 1 unmarked end, result remains stable.

Can extend to D^S by also allowing trees w/ $S \to \mathrm{Vert}(T)$ <u>not necess. injective</u>. ("$\overline{T^S}$").

(then adjacency also allows contracting edges b/w )

<u>Ex</u>: for $|S| = 2$, $T^S \longleftrightarrow$



ie. get $S^1 (\sim D^2 - D)$!

- Instead of stable trees, can consider all trees. They form a category $\mathcal{T}$; $\mathcal{T} \xrightarrow{\mathrm{Vert}} \mathrm{Sets}$; $\widetilde{\mathcal{T}} \longrightarrow \mathcal{T}$ category of pairs (tree, vertex of T)

$\widetilde{\mathcal{T}}^S = \widetilde{\mathcal{T}} \times_{\mathcal{T}} \cdots \times_{\mathcal{T}} \widetilde{\mathcal{T}} \setminus \mathrm{diag}.$ ($|S|$ times)