Dimensions of Triangulated Categories, joint work with M. Ballard and L. Katzarkov

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For two subcategories $\mathcal{I}_1$ and $\mathcal{I}_2$ we denote by $\mathcal{I}_1 * \mathcal{I}_2$ the full subcategory of objects $X \in \mathcal{T}$ such that there is a distinguished triangle $X_1 \to X \to X_2 \to X_1[1]$ with $X_i \in \mathcal{I}_i$.

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Further set $\mathcal{I}_1 \diamond \mathcal{I}_2 = \langle \mathcal{I}_1 * \mathcal{I}_2 \rangle$. By setting $\langle \mathcal{I} \rangle_1 := \langle \mathcal{I} \rangle$ we are able to inductively define $\langle \mathcal{I} \rangle_n := \langle \mathcal{I} \rangle_{n-1} \diamond \langle \mathcal{I} \rangle$.

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Let X be an object in $\mathcal{T}$. The generation time of X, denoted $\clock(X)$, is

$$\clock(X) := \min \{ n \in \mathbb{N} \,|\, \mathcal{T} = \langle X \rangle_{n+1} \}.$$

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The dimension of a triangulated category $\mathcal{T}$ is the minimal generation time among the strong generators i.e. the smallest number in the spectrum.

Results of Rouquier and Orlov

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2. if X is a smooth quasi-projective variety, then $\dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) \leq 2 \dim X$.

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Theorem (Orlov)

The above conjecture holds for smooth curves. More generally, if C is a smooth curve, then the spectrum of $\mathrm{D}^b(C)$ contains $\{1, 2\}$ with equality if and only if $C = \mathbb{P}^1$.

Tilting Objects and Dimensions of Derived Categories

Definition

Let $\mathcal{T}$ be a k-linear triangulated category. An object, T, of $\mathcal{T}$ is called a tilting object if the following two conditions hold:

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Proposition

Let T be a tilting object in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$, where X is smooth, and set $A := \mathrm{End}_X(T)$. Then the functors $\mathbf{R}\mathrm{Hom}_X(T, \bullet)$ and $\bullet \otimes_A T$ define exact equivalences between $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$ and $\mathrm{D}_{\mathrm{perf}}(A)$.

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Theorem (Christensen)

Let A be a coherent k-algebra and view it as an object of $\mathrm{D}^{\mathrm{b}}(\text{mod-}A)$. The generation time of A is the global dimension of A i.e. $\circleddash(A) = \mathrm{gd}(A)$.

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Suppose X is a smooth variety and T is a tilting object in $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. Let i_0 be the largest i for which $\mathrm{Hom}_X(T, T \otimes \omega_X^{\vee}[i])$ is nonzero. The global dimension of $\mathrm{End}_X(T)$ is bounded above by $\dim(X) + i_0$. Equality holds when X is proper over a perfect field.

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Corollary

Suppose X is a smooth variety and T is a tilting object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X)$. If $\mathrm{Hom}_X(T, T \otimes \omega_X^{\vee}[i]) = 0$ for all $i > 0$. Then $\dim \mathrm{D}^{\mathrm{b}}_{\mathrm{coh}}(X) = \dim(X)$.

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The Ghost Lemma and Generation Time

Ghost Lemma (simplified version)

Let X_i be objects of $\mathcal{T}$ for $1 \leq i \leq n$ and suppose we have a sequence of maps:

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-2}} X_{n-1} \xrightarrow{f_{n-1}} X_n,$$

such that the composition $f_{n-1} \circ \cdots \circ f_1$ is nonzero and $\mathrm{Hom}_{\mathcal{T}}(G[j], \bullet)(f_i) = 0$ for all i, j. Then $X_1 \notin \langle G \rangle_n$

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Example

Let X be a smooth affine variety of dimension n and x be a closed point. Let $\psi_1, \cdots \psi_n$ be a basis for $\mathrm{Ext}^1(k(x), k(x))$.

$$k(x) \xrightarrow{\psi_1} k(x)[1] \xrightarrow{\psi_2[1]} \cdots \xrightarrow{\psi_n[n-1]} k(x)[n]$$

Therefore, $k(x) \notin \langle \mathcal{O}_X \rangle_{n+1}$.

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Corollary

The generation time of a strong generator is precisely the maximal length of a sequence of “ghost” maps.

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When an exceptional collection, $\mathcal{A}$ is full and strong then $G_{\mathcal{A}}$ is a tilting object. Hence we have already described how one calculates the generation time of a full strong exceptional collection. We will soon provide another method of calculating the generation time of an exceptional collection which is not necessarily strong. But first we want to study mutations of exceptional collection and their relationship with generation time.

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Let $\mathcal{A} = \langle A_1, ..., A_n \rangle$ be a full exceptional collection for a k-linear triangulated category. For $i < j$, we define the a new exceptional object $L_i A_j$ by the following triangle:

$$\oplus_s \mathrm{Ext}^s(A_i, A_j) \otimes_k A_i[-s] \to A_j \to L_i A_j[1]$$

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and new exceptional collections by:

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$$\mathbf{R}_i \mathcal{A} = \langle A_1, ..., A_{i-2}, A_i, \mathrm{R}_i A_{i-1}, A_{i+1}, ..., A_n \rangle.$$

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Proposition

Suppose $\mathcal{A} = \langle A_1, ..., A_n\rangle$ and $\mathcal{B} = \langle B_1, ..., B_n\rangle$ are obtained from one another by a sequence of mutations. Let $⌚(G_{\mathcal{A}}) = s$ and $⌚(G_{\mathcal{B}}) = t$. Then the integer interval with endpoints s and t is contained in the spectrum.

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In particular, notice that for a full exceptional collection, $\mathcal{A} = \langle A_1, ..., A_n\rangle$, the dual collection is obtained by mutations as follows:

$$\mathcal{B} = (\mathbf{R}_n...\mathbf{R}_2)(\mathbf{R}_n...\mathbf{R}_3)(\mathbf{R}_n\mathbf{R}_{n-1})\mathbf{R}_n(\mathcal{A}).$$

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By these orthogonality relations, any non-isomorphism $B_{j_1} \to B_{j_2}$ is a ghost map for $G_{\mathcal{A}}$. It follows that a sequence of morphisms:

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Theorem

$\mathcal{A} = \langle A_1, ..., A_n \rangle$ be a full exceptional collection for a k-linear triangulated category. Consider the graded algebra $R = \boldsymbol{R}End(G_{\mathcal{A}})$. Then the generation time of $G_{\mathcal{B}}$ is bounded below by one less than the order the nilradical of $\boldsymbol{R}End(G_{\mathcal{A}})$. Furthermore, if $\boldsymbol{R}End(G_{\mathcal{A}})$ is formal, then this is the equality.

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Example + ϵ

On an algebraic variety, take a sequence of iterated blow-ups, where one continues to blow up along the newly introduced exceptional divisor. One can obtain arbitrarily high generation times depending on how many blow ups one takes.

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Putting everything together we get the following:

Corollary

Suppose $\mathcal{A}$ is a full strong exceptional collection in a triangulated category $\mathcal{T}$. Let r be the projective dimension of $\mathrm{End}(G_{\mathcal{A}})$ and s be the the Lowey length (longest sequence of arrows in the corresponding quiver). Then the integer interval $\{r, \ldots, s\}$ or $\{s, \ldots, r\}$ is contained in the dimension spectrum of $\mathcal{T}$.

Generation Time for Exceptional Collections

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Example + ϵ

Let Q be a quiver such that the underlying graph is a Dynkin diagram of type A_n. The dimension spectrum of $\mathrm{D}^{\mathrm{b}}_{\mathrm{f.g.}}(kQ)$ is equal to the integer interval $\{0, ..., n-1\}$. This can also be interpreted as the derived Fukaya category of an A_n singularity.

Semiorthogonal Decompositions

Definition

A semiorthogonal decomposition of a triangulated category, $\mathcal{T}$, is a sequence of admissable triangulated subcategories $\mathcal{A}_1, ..., \mathcal{A}_n$ such that there are no morphisms from any object in $\mathcal{A}_i$ to any object in A_j for $i > j$ and $\mathcal{T}$ is generated by these subcategories.

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Notice that a full exceptional collection is a semiorthogonal decomposition where all the components consist of one object up to sums and shift. In particular all the components have spectrum $\{0\}$.

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Suppose $\langle \mathcal{A}_1, ..., \mathcal{A}_n \rangle$ and $\langle \mathcal{B}_1, ..., \mathcal{B}_n \rangle$ are semiorthognal decomposition of $\mathcal{T}$ which are mutants of one another. Let $G_1 \oplus .. \oplus G_n$ be a generator with $G_i \in \mathcal{A}_i$ and $H_1 \oplus ... \oplus H_n$ a mutated generator. Let $M = \max_i \{ ⏲(G_i) \}$, $⏲(G_1 \oplus .. \oplus G_n) = s$, and $⏲(H_1 \oplus ... \oplus H_n) = t$. Then the largest possible gap in the spectrum on the interval containing s and t is M. In particular, if all the G_i generate as quickly as possible, then the maximal gap is the maximal dimension achieved by one of the $\mathcal{A}_i$.

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Notice that if $\mathcal{T}$ is the derived category of an algebraic variety, then by blowing up, one introduces new components to the semiorthogonal decomposition which conjecturally live in codimension 2. One may hope that this introduces gaps in the spectrum which are most codimension 2.

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Example

Let X be a cubic 3-fold. Then we have a semiorthgonal decomposition $\langle \mathcal{A}_X, \mathcal{O}, \mathcal{O}(1) \rangle$. One can show that the dimension of $\mathcal{A}_X$ is at least 2.

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The classical proof that X is not rational is by means of the Griffiths-Clemens component of X. The category $\mathcal{A}_X$ in this case is the analogue of that component.

The Dimension Spectrum of an Elliptic Curve

As mentioned previously, the set $\{1, 2\}$ is a proper subset of the dimension spectrum of $\mathrm{D}^{\mathrm{b}}(C)$ where C is a proper, nonrational curve. For genus one we have:

Theorem

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This result is proven by first showing that the slowest generator is $\mathcal{O} \oplus \mathcal{O}_p$, that this particular generator has generation time 4, and that there also exists generators of generation time $1, 2$, and 3. For us, the interesting part is the ghost sequence which gives a lower bound of 4 on the generation time of $\mathcal{O} \oplus \mathcal{O}_p$.

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Take p to be the identity and let q be a point of order two. We have the following ghost-sequence:

$$\mathcal{O}_q \to \mathcal{O}(-q)[1] \to \mathcal{O}(p-q)[1] \to \mathcal{O}(2p-q)[1] \to \mathcal{O}_q[1].$$

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Notice that $\mathcal{O}$ and $\mathcal{O}_p$ form an A_2 configuration of spherical objects. Let T_A be the spherical twist by $\mathcal{O}$ and T_B be the spherical twist by $\mathcal{O}_p$. This ghost sequence can then be interpreted as:

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Also notice that $\mathcal{O}_q$ is orthogonal to $\mathcal{O}_p$ and that the third term $\mathcal{O}(p-q)$ is orthogonal to $\mathcal{O}$. As it turns out, the above is a general phenomenon involving braid group actions and generalizes nicely to derived Fukaya Categories of curves.

The Dimension Spectrum of the Derived Fukaya Category of a Curve

On a curve, C, of genus g there is an A_{2g} configuration of spherical objects which generates the derived Fukaya category, these curves can be chosen to be anti-invariant with respect to the hyperelliptic involution. Denote the objects by $S_1, ..., S_{2g}$ and the corresponding spherical twists by $s_1, ..., s_n$. In analogy with the elliptic curve, consider a loop X which is orthogonal to $S_2, ..., S_{2g}$ and invariant under the hyperelliptic involution.

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Then the following sequence is a ghost sequence for $G = S_1 \oplus ... \oplus S_{2g}$:

$$X \to s_1(X) \to ... \to s_{2g}...s_1(X) \to ... \to s_1...s_{2g}s_{2g}...s_1(X).$$

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In general, one can prove:

Theorem

Let G be the generator described above on a curve of genus g. Then $4g \leq \clock(G) \leq 4g+2$.

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The upper bound comes from the following relation, expressing a deformation of the Serre functor as a product of spherical twists:

$$(s_1...s_{2g})^{2g+1} = \tau[1]$$

The Dimension Spectrum of the Derived Fukaya Category of a Curve

Example

Consider an iterated blow up of $\mathbb{P}^2$ at a point n-times where one continues to blow up the newly formed exceptional divisor. Using exceptional objects, one can form a generator with generation time $n + 2$. However, there also exists an A_{n-1} configuration of spherical objects. This configuration takes at least $2n - 3$ steps and in all likely hood one can get a $4 + 2n$ step generator in essentially this way. Combining generators which use spherical and exceptional objects, should thus provide a way of filling in the higher end of the spectrum, say from $n + 2$ to $4 + 2n$.