B. Toen – 20/1/10 – Saturated dg-categories
Part III: Topological invariants

$T\ (\simeq \widehat{T_0})$ $\underbrace{\text{compactly generated}}_{\Downarrow}$ dg-cat. (locally presentable)

T_0 = compact obj of T.

T has a dual in $\mathcal{D}_g^{lp}(k)$, $T^\vee = \widehat{T_0^{op}}$

$$\begin{tikzcd} \mathbb{1} \arrow[r, "id"] \arrow[rr, bend right, "Tr(id_T) = \text{``}\dim T\text{''}"'] & T \otimes T^\vee \arrow[r, "ev"] & \mathbb{1} \end{tikzcd}$$

$End(\mathbb{1}) = \hat{k}$ $((2,\infty)$-cat.$)$.

<u>Remark</u>: $Tr(id) \simeq HH_\bullet(T_0)$

There is a categorical reason for S^1 acting on $Tr(id)$

$\rightsquigarrow$ $HC_\bullet(T) \simeq HH_\bullet(T_0)_{S^1}$

$HC^\bullet(T) \simeq HH^\bullet(T_0)^{S^1}$

and $HP_\bullet(T) \simeq HC_\bullet(T)[u^2]$

$HP_\bullet(T)$ is a version of de Rham cohomology of T

(recall $HP_\bullet(D_{dg}(X)) \simeq H^\bullet_{dR}(X/k)$ for X finite type flat$/k$ char $=0$.)

$HP_\bullet(T)$ is close to being a topological cohom. invariant of T, <u>but</u> it's only defined over k, not over $\mathbb{Z}$ or $\mathbb{Q}$.

- <u>Goal</u>: <u>construct a cohom. theory well-defined over $\mathbb{Q}$ ($k=\mathbb{C}$) or at least $\mathbb{Q}_\ell$</u>

  When $k=\mathbb{C}$, this cohomology theory $/\mathbb{Q}$ should give $\mathbb{Q}$-structures on $HP_\bullet(T)$ when T saturated $/\mathbb{C}$.

- <u>Suggestion</u> (Neeman-Bondal):

  T dg-cat$/k$ $\rightsquigarrow$ consider $\mathcal{M}_T$ moduli of compact objects in T and consider its "geometric realization" $|\mathcal{M}_T|^{top}$ (usual one for $k=\mathbb{C}$ étale one in general)

Moduli functors $\mathcal{M}_T$: 2 definitions.

$$\mathcal{M}_T : (\mathrm{Aff}/k)^{op} \longrightarrow \{\infty\text{-groupoids}\} \xrightarrow{\sim} \{\text{simplicial sets}\} \simeq \{\mathrm{Top}\}$$

$$A/k \longmapsto \left\{ \begin{array}{l} \text{objects in } T \hat{\otimes}_k^{c} \hat{A} \\ \text{which are perfect}/A \end{array} \right\} \simeq \left(\begin{array}{l} \text{Nerve of the} \\ \text{category of } T_0^{op} \otimes_k A\text{-mod} \\ \text{perfect}/A \end{array} \right)$$

$$= \left\{ \begin{array}{l} E \,/\, \forall x \in T_c,\ ev_x(E) \\ \text{is a perfect complex}/A \end{array} \right\}$$

$$\mathcal{M}'_T : (\mathrm{Aff}/k)^{op} \longrightarrow \{\infty\text{-groupoids}\}$$

$$(A/k) \longmapsto \left\{ \begin{array}{l} \text{objects in } T \otimes_k^{ct} \hat{A} \\ \text{which are compact} \end{array} \right\} \simeq \left\{ \begin{array}{l} \text{nerve of compact} \\ T_0^{op} \otimes_k A\text{-modules} \end{array} \right\}$$

Facts:

1. T of finite type $\Rightarrow \mathcal{M}_T$ algebraic
2. T smooth $\Rightarrow \mathcal{M}_T \hookrightarrow \mathcal{M}'_T$
3. T proper $\Rightarrow \mathcal{M}'_T \hookrightarrow \mathcal{M}_T$
4. T saturated $\Rightarrow \mathcal{M}_T = \mathcal{M}'_T$

Geometric realizations: $(k = \mathbb{C})$

$$\begin{tikzcd} & \Pr(\mathrm{Aff}^{ft}/\mathbb{C}) \arrow[d, "\exists !"] \\ (\mathrm{Aff}^{ft}/\mathbb{C}) \arrow[ur, hook, "h"] \arrow[r, "1-1"] & \mathrm{Top} \end{tikzcd}$$

presheaves of sets

$\exists !$ s.t. commutes with colimits

$$(\operatorname{Spec} A) \longmapsto ((\operatorname{Spec} A)(\mathbb{C}))^{top} = \underline{\mathrm{Hom}}(A, \mathbb{C})^{an}$$

$F \in \Pr(\mathrm{Aff}^{ft}/\mathbb{C})$ then $|F| = \underset{\operatorname{Spec} A \to F}{\underrightarrow{\mathrm{hocolim}}} |\operatorname{Spec} A|$.

Important property: $F \mapsto |F|$ sends local isomorphisms to weak equivalences (Van Kampen)

This also works for presheaves of ∞-groupoids:

$$\begin{tikzcd} & \infty\text{-}\Pr(\mathrm{Aff}^{ft}/\mathbb{C}) \arrow[d, "\exists !"', "1.9B"] \\ (\mathrm{Aff}^{ft}/\mathbb{C}) \arrow[ur, hook] \arrow[r] & \mathrm{Top} \end{tikzcd}$$

(Betti)

Def: T compactly generated loc. pres. dg cat$/\mathbb{C}$.

- The semitopological K-theory of T is $K^{\bullet}_{semi}(T) = |\mathcal{M}_T|^B \in Top$
- The semitopological K-homology of T is $K_{\bullet}^{semi}(T) = |\mathcal{M}'_T|^B \in Top$.

Important remark: $K^{\bullet}_{semi}(T)$ and $K_{\bullet}^{semi}(T)$ are spectra.

The "group" structure is induced by direct sum of objects in T.

Moreover: $K^{\bullet}_{semi}(T)$ & $K_{\bullet}^{semi}(T)$ are modules over $K^{\bullet}_{semi}(\hat{\mathbb{C}})$ — commutative ring spectrum

$K^{\bullet}_{semi}(\hat{\mathbb{C}})$ acts by external tensor product.

Key proposition: $\exists$ natural equivalence of ring spectra

$k^{top} = bu[\beta^{-1}]$, β Bott element

$$\begin{tikzcd} K^{\bullet}_{semi}(\hat{\mathbb{C}}) & bu \arrow[l, "\sim"'] & \simeq \mathbb{Z} \times BU \\ & \left|\coprod_n BGl_n\right|^+ \arrow[ul] \arrow[u, "\simeq"'] & \end{tikzcd}$$

Def: T c.g. l.p. dg cat$/\mathbb{C}$. The top. K-theory of T is:

$$K^{\bullet}_{top}(T) = K^{\bullet}_{semi}(T) \wedge_{bu} K^{top} = K^{\bullet}_{semi}(T)[\beta^{-1}]$$

$$K_{\bullet}^{top}(T) = K_{\bullet}^{semi}(T)[\beta^{-1}]$$

Conjectures:

1) T of finite type $\Rightarrow$ $K^{\bullet}_{top}(T)$ is a bu-module of finite type.

2) T of finite type $\Rightarrow$ $K^{\bullet}_{top}(T) \otimes_{\mathbb{Z}} \mathbb{C} \xrightarrow{\sim} HP_{\bullet}(T)$ — Chen character.