B. Toën – 18/1/10 – Saturated dg-categories I

Goal: present some recent results on saturated dg-algebras.

Origins: 1) X smooth proper scheme $/k$

$D_{perf}(X)$ has a generator E;

$B := R\underline{End}(E)$ dg-algebra$/k$.

($:= \underline{End}(E')$, E' injective replacement of E)

Fact:
- $D_{perf}(X) \simeq D_{perf}(B\text{-dgmod})$
- B is saturated

2) X CW-complex, connected; $\Omega_* X$ based loops (topological group)

$C_*(\Omega_* X, k) =: B$ dg-algebra$/k$

(Multiplication in B is induced by $\Omega_* X \times \Omega_* X \to \Omega_* X$).

When X is a finite CW-complex, B is smooth ($\frac{1}{2}$ saturated)

- $D(B\text{-dgmod}) \simeq D_{loc}(X,k) \subseteq D(X,k)$

$$D_{loc}(X,k) = \{E \,/\, \underline{H}^i(E) \text{ is locally constant } \forall i\}$$

- Hence, in both cases, $D(X) \simeq D(B)$, and many interesting invts of X can be recovered from B.

Ex:
- X smooth proper scheme $/k$, $\mathrm{char}(k) = 0$

  $\Rightarrow HH_\bullet(B/k) \simeq \bigoplus H^\bullet(X, \Omega^\bullet_{X/k})$
- X finite CW-complex $\Rightarrow HH_\bullet(B/k) \simeq H_\bullet(\mathcal{L}X)$ (free loop space)

Part I – Finiteness results about dg-algebras

Notations: k comm. ring, B dg-algebra$/k$ (associative, with unit)

$D(B)$ = derived cat. of dg-modules over B

Def: B is <u>proper</u> if B is a perfect complex of k-modules ($\Leftrightarrow$ B is compact in $D(k)$) (ie. $B \underset{q\text{-isom.}}{\sim}$ bounded complex of proj. k-modules of finite type)

Rmk: if k is a field, B proper $\Leftrightarrow$ $H^\bullet(B)$ is finite dimensional.

Def: B is smooth if B is compact in $D(B \overset{L}{\underset{k}{\otimes}} B^{op})$ (dg bimodules/B)
ie. $[B, \oplus -] = \oplus [B, -]$
B is saturated if it is proper & smooth

Def: B is of finite type if $\forall$ filtered system of dg-algebras $\{B_\alpha\}$,
$[B, \operatorname{colim}_\alpha B_\alpha] \simeq \operatorname{colim}_\alpha [B, B_\alpha]$

where $[B,C] :=$ set of morphisms $B \to C$ in $H_0(dg\,alg/k)$
$H_0(dg\,alg/k) := (\text{quasi-isom})^{-1}(dg\,alg/k)$

Prop:
- finite type $\Rightarrow$ smooth
- saturated $\Rightarrow$ finite type.

Many smooth dg algs "occuring naturally" are of finite type
(counterexamples: e.g. $\mathbb{A}^1 \smallsetminus \infty$ many pts)

Examples:

1) X scheme, flat, of finite type $/k$ (separated)
B dg-alg. q-iso to endomorphisms of a compact generator,
so $D_{perf}(X) \simeq D(B)$
Then: B proper $\Leftrightarrow$ X proper
B smooth$/k$ $\Leftrightarrow$ X smooth$/k$
B finite type $\Leftrightarrow$ X smooth$/k$

2) X finite CW complex, $D_{loc}(X, k) \simeq D(B)$ for $B = C_*(\Omega X)$
$\Rightarrow$ B is of finite type (hence smooth)

3) Q finite quiver, $B = A(Q,k)$ path algebra is of finite type$/k$.

Rmk: As we'll see tomorrow, these notions (smooth, proper, finite type) have a nice categorical interpretation in terms of duality in a certain $\otimes$-2-category of dg-categories.
($\leftrightarrow$ dualizability, half-dualizability).

Main theorems:

Thm 1: If B is a finite type dg-algebra/k then there exists an algebraic moduli space for finite dim$^!$ B-dg modules

Thm 2: There is a moduli space for saturated dg-algebras (up to quasi-isom.)

Rmk: what are these moduli spaces?

- they should at least be algebraic stacks, because objects have automorphisms.
- moreover, there are "higher automorphism groups" (in both cases).

E.g.: E dg mod/B $(E \in D(B)) \Rightarrow$

- $Aut(E)$ automorphisms
- but also: we have homotopies b/w Aut's, in particular
  $Ext^{-1}(E,E)$ = self-homotopies of id_E
  = "2-automorphism group" (unit = 0-homotopy)
  ie. $Aut(E)$ can't be an algebraic group, it must be stacky itself!
- and so on! $Ext^{1-i}(E,E)$ i-automorphism group (self-homotopies of $id_{id_{\dots id_E}}$)

$\Rightarrow$ moduli spaces must be algebraic ∞-stacks similarly for thm. 2, e.g. self-homotopies of id_B = derivations of B

(works up to quasi-iso., not up to Morita equivalence, hence deform^n theory is governed by derivation complex, not $HH^{\bullet}$)

$B \in H_0(dgalg/k) \rightsquigarrow$ $aut(B)$ automorphism gp of B

$Der^{-1}(B,B)$ self-homotopies of id_B

$\dots Der^{1-i}(B,B)$

where $Der^i(B,M) = Ext^i_{D(B^{op} \otimes B)}(I_B, M)$

for $M \in D(B^{op} \otimes B)$ and $0 \to I_B \to B^{op} \otimes B \to B \to 0$.

(differs from $HH^{\bullet}(B)$ by a factor of $H^{\bullet}(B)$).

[NB: thm 2 would be false for dg algs. up to Morita equiv^ce, that moduli space is at most a <u>formal</u> stack, <u>not</u> alg.].

<u>Consequences:</u>

1. • up to stratification (inductive construction), we get schemes of finite type /k classifying saturated dg-algebras, and finite dim. dg modules over a finite type dg algebra. (+ $aut(E)$, $Ext^{1-i}(E,E)$, ... are flat group schemes locally defined on these schemes).

2. B dg alg of finite type $\Rightarrow \exists$ algebraic space M_B, locally of finite type /k, classifying <u>simple objects</u>

$M_B \longleftrightarrow \{E \in D(B) \,/\, \text{finite dim.}, Ext^{-i}(E,E)=0, Ext^0(E,E) \simeq k\}$

M_B comes equipped with a natural class $\alpha \in H^2_{ét}(M_B, \mathbb{G}_m)$, obstruction for the existence of a universal simple dg-module over M_B.

3. X compact complex manifold, if $D^b_{coh}(X) \simeq D_{perf}(B)$ with B saturated, then X is algebraic.

(embed X into M_B by considering skyscraper sheaves).