B. Keller – 22/1/10 – The periodicity conjecture via CY categories

periodicity conjecture (Zamolodchikov 1991) – math physics

We'll explain a proof based on homological algebra (2-CY tri. cats.)

<u>Ideas</u>: • <u>categorification</u>, i.e. periodicity conj. = combinatorial shadow of a phenomenon about tri. categories

• <u>cluster algebras</u> (Fomin–Zelevinsky) form the interface b/w categorical setup & combinatorial conjecture

<u>Plan</u>: 1) The conjecture

2) The beginning of the proof: categorification of root systems

3) The end of the proof: homological periodicity

4) Dessert: quiver version of the conjecture

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1. <u>The conjecture</u>

Δ, Δ' Dynkin diagrams (simply laced)

vertices $I = \{1, \dots, n\}$, $I' = \{1, \dots, n'\}$

Coxeter numbers: h, resp. h', cf. table

Incidence matrices: A, A' $\quad a_{ij} = \begin{cases} 1 & \text{if } \overset{i}{\bullet}\!\!-\!\!\!-\!\!\overset{j}{\bullet} \\ 0 & \text{otherwise} \end{cases}$

Δ	h
A_n	$n+1$
D_n	$2n-2$
E_6	12
E_7	18
E_8	32

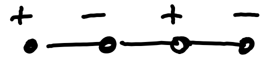
<u>Associated Y-system</u>:

$$\begin{cases} \text{variables } y_{i,i',t} \quad i \in I,\ i' \in I',\ t \in \mathbb{Z} \\ \text{equations: } \quad y_{i,i',t-1}\, y_{i,i',t+1} = \dfrac{\prod_{j=1}^{n} (1 + y_{j,i',t})^{a_{ij}}}{\prod_{j'=1}^{n'} (1 + y_{i,j',t}^{-1})^{a'_{i'j'}}} \end{cases}$$

<u>Conj</u>: All solutions of this system are periodic with period dividing $2(h+h')$.

<u>Algebraic reformulation</u>: $\mathcal{F} = \mathbb{Q}(y_{i,i'} \mid i \in I, i' \in I')$

Choose $\eta : I \to \{\pm 1\}$, $\eta' : I' \to \{\pm 1\}$ s.t. adjacent vertices have diff^t signs

E.g. for $\Delta = A_4$, 

For $\varepsilon \in \{\pm 1\}$, define $\tau_\varepsilon : \mathcal{F} \xrightarrow{\sim} \mathcal{F}$ by

$$\tau_\varepsilon(y_{i,i'}) = \begin{cases} 1/y_{ii'} \cdot \prod_{j=1}^{n} (1+y_{ji'})^{a_{ij}} \Big/ \prod_{j'=1}^{n'} (1+y_{ij'}^{-1})^{a'_{i'j'}} & \text{if } \varepsilon = \eta(i)\eta'(i') \\ y_{i,i'} & \text{otherwise} \end{cases}$$

$$\varphi_{Zam} = \tau_+ \circ \tau_- : \mathcal{F} \xrightarrow{\sim} \mathcal{F}$$

Then <u>Conj'</u>: $\Big\Vert\ \varphi_{Zam}^{h+h'} = Id_{\mathcal{F}}$

<u>History:</u> Conj. was stated for
Δ, A_1 : A. Zamolodchikov, 1991
Δ, A_n : Kuniba-Nakanishi, 1992
Δ, Δ' : Ravanini-Valleriani-Toledo, 1992

was proved for
A_n, A_1 : Frenkel-Szenes '95, Gliozzi-Tateo '96
Δ, A_1 : Fomin-Zelevinsky 2003
A_n, A_m : Volkov, Szenes, Henriques 2007

<u>Thm:</u> $\Big\Vert$ The conj. is true in general.

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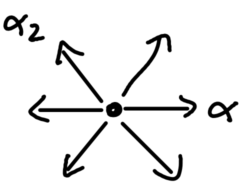
2. <u>The beginning of the proof: categorification of root systems</u>

Δ simply laced Dynkin diagram

- $\alpha_1 \dots \alpha_n$ simple roots
- s_{α_i} = reflection at $(\mathbb{R}\alpha_i)^\perp$
- $c = s_{\alpha_1} \dots s_{\alpha_n}$ Coxeter element
- h = order of c

<u>Ex.:</u> $\Delta = A_2$



c = rot. by $120°$
$h = 3$

Categorify these as follows:

- Q quiver with underlying graph Δ

  $\mathbb{C}Q$ path algebra

  $\mathrm{mod}(\mathbb{C}Q)$ = finite dim. $\mathbb{C}Q$-right modules $= \mathrm{rep}_{\mathbb{C}}(Q^{op})$

  $\mathcal{D}_Q = D^b(\mathrm{mod}\text{-}\mathbb{C}Q)$ triangulated cat.

  <u>Notation</u>: $[1]$ = suspension functor $L \mapsto L[1]$

  S = Serre functor $= - \otimes_{\mathbb{C}Q} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}Q, \mathbb{C})$

  $\mathrm{Hom}_{\mathcal{D}_Q}(X,Y)^* \simeq \mathrm{Hom}_{\mathcal{D}_Q}(Y, SX)$ $\forall X, Y$, bifunctorially

- <u>Thm</u>: (Gabriel, Happel)

  $\exists$ canonical isom. $K_0(\mathcal{D}_Q) \xrightarrow{\sim} \bigoplus_{i=1}^{n} \mathbb{Z}\alpha_i$

  $[S_i] \longmapsto \alpha_i$

  simple module at vertex i

  $\{[X] / X \text{ indecomposable}\} \xrightarrow{\sim} \{\text{roots}\}$

  Auslander-Reiten translation functor $= \tau^{-1} = S^{-1}[1]$ $\quad\circlearrowleft$ $\quad\circlearrowleft_c$ Coxeter element

  $\tau^{-h} \simeq [2] \longleftrightarrow c^h = \mathrm{Id}$

This provides a categorification of root systems.

<u>Cluster algebras</u> (of finite type) are a refinement of root systems. (Fomin-Zelevinsky)

They can also be categorified.

<u>Thm</u>: (Buan-Marsh-Reineke-Reiten-Todorov):

The cluster algebra $\mathcal{A}_\Delta$ is "categorified" by the cluster category $\mathcal{C}_Q :=$ orbit (quotient) category $\mathcal{D}_Q / (S^{-1}[2])$

objects = same as for $\mathcal{D}_Q$

$\mathrm{Hom}_{\mathcal{C}_Q}(X,Y) = \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{D}_Q}(X, S^{-p}Y[2p])$

Rmk: can show $\mathcal{C}_Q$ is hom-finite and triangulated (not automatic !!)

Also, $\mathcal{C}_Q$ is clearly 2-CY

(by construction: $/S^{-1}[2]$ gives largest quotient s.t. $S^{-1}[2] \simeq id$).

Skip: middle of the proof

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3. End of the proof: categorical periodicity:

Δ, Δ' Dynkin diagrams, Q, Q' orientations of Δ, Δ'

$\mathcal{D}_{Q,Q'}$ = bounded der. cat. of modules over $(\mathbb{C}Q \underset{\mathbb{C}}{\otimes} \mathbb{C}Q')$ (has gldim ≤ 2).

$$\underbrace{\mathcal{D}_{Q,Q'}/S^{-1}[2]}_{\text{not triang. in general}} \hookrightarrow \mathcal{C}_{Q,Q'} \quad \text{triangulated hull, } \underline{\underline{\text{still 2-CY!}}}$$

Recall: we'd like to categorify $\varphi_{\mathrm{Zam}}: \mathcal{F} \xrightarrow{\sim} \mathcal{F}$

Def. $\Phi_{\mathrm{Zam}} := \underset{\underset{S^{-1}_{\mathcal{D}_Q}[1]\otimes \mathbb{1}}{\shortparallel}}{\tau^{-1} \otimes \mathbb{1}} : \mathcal{C}_{Q,Q'} \xrightarrow{\sim} \mathcal{C}_{Q,Q'}$.

Main thm 2.
a) $\Phi_{\mathrm{Zam}}^{h+h'} \simeq Id$
b) Φ_{Zam} "categorifies" φ_{Zam}.

Proof of a): $S \otimes S \underset{(\mathcal{D}_{Q,Q'} \cong \mathcal{D}_Q \otimes \mathcal{D}_{Q'})}{\xrightarrow{\sim}} S_{\mathcal{D}_{Q,Q'}} \underset{(\mathcal{C}_{Q,Q'} \text{ is 2-CY})}{\simeq} [2] \simeq [1] \otimes [1]$.

So $S[-1] \otimes S[-1] = \tau \otimes \tau \simeq \mathbb{1} \Rightarrow \tau^{-1} \otimes \mathbb{1} \simeq \mathbb{1} \otimes \tau$.

(ie: Φ_{Zam} more symmetric than it looks!).

and $\Phi_{Zam}^{h+h'} = (\tau^{-1}\otimes \mathbb{1})^h(\mathbb{1}\otimes \tau)^{h'} \simeq ([2]\otimes\mathbb{1})(\mathbb{1}\otimes[-2]) = \mathbb{1}$

Gabriel-Happel

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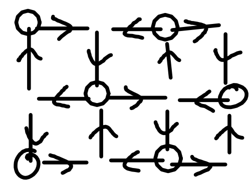
4. Quiver version of the conj.

Δ, Δ' simply laced, R = quiver obtained from $\Delta\times\Delta'$ by choosing locally cyclic orientation. (ie. loc. $\circlearrowleft$)

& color vertices of R alternatingly white ($\circ$) or black.

E.g: $(A_4, A_3) \rightsquigarrow R =$ 

$$\varphi_{quiver} = \Big(\prod_{i\ \text{black}} \mu_i\Big)\Big(\prod_{i\ \text{white}} \mu_i\Big) \qquad (\mu_i = \text{Fomin-Zelevinsky mutations})$$

(no edges b/w same color $\Rightarrow$ μ_i's in $\prod_{\text{color}}$ commute)

Easy: $\varphi(R) = R$, ie. φ autom. of R in mutation groupoid.

Main thm 3: $\varphi^{h+h'}(\tilde R) = \tilde R$ for any overquiver $\tilde R \underset{\text{full}}{\supset} R$.

This is equivalent to the conj, but unlike it, it can be checked efficiently even on large examples [mutation applet].

NB: $\tilde R = \text{stuff} \rightleftarrows R$; φ preserves stuff & R but affects arrows between them

Rmk: quiver mutation $\equiv$ how Ext^1-quiver of exc. coll. in a 3CY category changes under tilting