A. Kuznetsov - 22/1/10 - HH_* and HH^* of admissible subcategories

- <u>Recall:</u> for scheme X, X smooth proj.

$$\begin{cases} HH^\bullet(X) = \mathrm{Hom}^\bullet_{X\times X}(\underbrace{\Delta_*\mathcal{O}_X}_{\text{diagonal}}, \Delta_*\mathcal{O}_X) \\ HH_\bullet(X) = H^\bullet(X\times X, \Delta_*\mathcal{O}_X \overset{L}{\otimes} \Delta_*\mathcal{O}_X) \\ \qquad \simeq \mathrm{Hom}^\bullet_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\omega_X[\dim X]) \end{cases}$$

B dg-algebra $\Rightarrow$ $\begin{cases} HH^\bullet(B) = \mathrm{Hom}_{B\otimes B^{op}}(B, B) \\ HH_\bullet(B) = B \overset{L}{\underset{B\otimes B^{op}}{\otimes}} B \end{cases}$

If $\mathcal{E}$ strong generator for $D^b(X)$, $B := R\mathrm{Hom}^\bullet(\mathcal{E},\mathcal{E})$

$$\Rightarrow HH^\bullet(X) \simeq HH^\bullet(B), \quad HH_\bullet(X) \simeq HH_\bullet(B)$$

Moreover $HH_d(X) = \bigoplus_p H^{p+t}(X, \Omega^p_X)$ and $HH^d(X) = \bigoplus_p H^{t-p}(X, \Lambda^p T_X)$

- <u>Semiorthogonal decompn</u> of $\mathcal{T}$:= $\mathcal{A}_1 \dots \mathcal{A}_m$ full tri subcats st.
  1) $\mathrm{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0 \ \forall j < i$
  2) $\forall T \in \mathcal{T}$, $\exists\, 0 = T_m \to T_{m-1} \to \dots \to T_1 \to T_0 = T$,
  st $\mathrm{Cone}(T_i \to T_{i-1}) \in \mathcal{A}_i$

  Write $\mathcal{T} = \langle \mathcal{A}_1 \dots \mathcal{A}_m \rangle$.

- $\mathcal{A} \overset{\alpha}{\hookrightarrow} \mathcal{T}$ is <u>admissible</u> if α has both left & right adjoints.
  If $\mathcal{A} \subset \mathcal{T}$ is admissible then $\mathcal{T} = \langle \mathcal{A}, {}^\perp\mathcal{A} \rangle = \langle \mathcal{A}^\perp, \mathcal{A} \rangle$.

Let $\mathcal{A} \subset D^b(X)$ admissible, $\mathcal{E}$ strong gen. for $D^b(X)$.

$\mathcal{E}_\mathcal{A}$ = the component of $\mathcal{E}$ in $\mathcal{A}$ (ie. piece of filtration of $\mathcal{E}$ in)

$B_\mathcal{A} = R\mathrm{Hom}^\bullet(\mathcal{E}_\mathcal{A}, \mathcal{E}_\mathcal{A})$

Then $HH^\bullet(\mathcal{A}) = HH^\bullet(B_\mathcal{A})$, $HH_\bullet(\mathcal{A}) = HH_\bullet(B_\mathcal{A})$.

Consider projection functor to $\mathcal{A}$, $\quad D^b(X) \longrightarrow D^b(X)$
$\qquad \mathcal{E} \longmapsto \mathcal{E}_{\mathcal{A}}$

Lemma: $\exists!\ P \in D^b(X \times X)$ s.t. projection functor is isom. to ϕ_P,
$\phi_P(\mathcal{E}) := p_{1*}(P \otimes p_2^* \mathcal{E})$.

Pf: $D^b(X) = \langle \mathcal{A}_1 \ldots \mathcal{A}_m \rangle \Rightarrow D^b(X \times X) = \langle \mathcal{A}_{1X} \ldots \mathcal{A}_{mX} \rangle$
$$\begin{matrix} \cup \\ \Delta_* \mathcal{O}_X \end{matrix}$$
P_i = the component of $\Delta_* \mathcal{O}_X$ in $\mathcal{A}_{iX}$ $\Rightarrow$ ϕ_{P_i} gives the proj$^{\underline{n}}$. ▲

Thm: $HH^\bullet(\mathcal{A}) \cong \mathrm{Hom}_{D^b(X \times X)}(P, P)$
$HH_\bullet(\mathcal{A}) \cong H^\bullet(X \times X, P \overset{L}{\otimes} P^T)$

Pf: $\mathcal{E}_{\mathcal{A}}$, $\mathcal{A} \cong D^{perf}(B_{\mathcal{A}})$
$$D^{perf}(B_{\mathcal{A}} \otimes B_{\mathcal{A}}^{op}) \xhookrightarrow[\text{fully faithful}]{} D^b(X \times X)$$
$$B_{\mathcal{A}} \longmapsto P$$
▲

Properties:

1) Functoriality of $HH_\bullet$: $K \in D^b(X \times Y)$, $\mathcal{A} \subset D^b(X)$, $\mathcal{B} \subset D^b(Y)$
s.t. $\phi_K(\mathcal{A}) \subset \mathcal{B}$ $\Rightarrow$ then $\phi_{K_*}: HH_\bullet(\mathcal{A}) \to HH_\bullet(\mathcal{B})$

2) $HH^\bullet$ is functorial wrt equivalences (but not wrt general functors)

- If $\mathcal{A} = \langle \mathcal{A}_1, \mathcal{A}_2 \rangle$ then $HH_\bullet(\mathcal{A}) = HH_\bullet(\mathcal{A}_1) \oplus HH_\bullet(\mathcal{A}_2)$.

$D^b(X) = \langle \mathcal{A}_1 \ldots \mathcal{A}_m \rangle$
$P_1 \ldots P_m$ projection kernels
$\phi_{P_1} \ldots \phi_{P_m}$ proj. functors
then $HH_\bullet(\mathcal{A}_i) \cong \mathrm{Im}\, \phi_{P_{i*}} \subset HH_\bullet(X)$

- $\underset{P}{\mathcal{A}} = \langle \underset{P_1}{\mathcal{A}_1}, \underset{P_2}{\mathcal{A}_2} \rangle$ $\Rightarrow$ then $\exists$ long exact seq^s:

$$\cdots \to HH^d(\mathcal{A}) \to HH^d(\mathcal{A}_1) \oplus HH^d(\mathcal{A}_2) \to \mathrm{Hom}^{d+1}(P_1, P_2) \to HH^{d+1}(\mathcal{A}) \to \cdots$$

$$\cdots \to HH^d(\mathcal{A}) \to HH^d(\mathcal{A}_1) \to \mathrm{Hom}^{d+1}(P_2', P_2) \to HH^{d+1}(\mathcal{A}) \to \cdots$$

where P_2' = proj. kernel for $\mathcal{A} = \langle \mathcal{A}_2, {}^\perp\mathcal{A}_2 \rangle$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad P_2'$

__Rmk:__ $\mathrm{Hom}^{d+1}(P_1, P_2) \simeq \mathrm{Hom}^d(\phi, \phi)$ with $\phi =$ __gluing functor__ $= \alpha_2^* \circ \alpha_1$

__Q:__ what about $\mathrm{Hom}^{d+1}(P_2', P_2)$? $\qquad \mathcal{A}_1 \underset{\alpha_1}{\subset} \mathcal{A} \underset{\alpha_2}{\supset} \mathcal{A}_2$

__Examples:__

1) Assume $\mathcal{O}_X$ is exceptional $\left(H^p(X, \mathcal{O}_X) = \begin{cases} k & p=0 \\ 0 & \text{else} \end{cases} \right)$

Then $D^b(X) = \langle \mathcal{A}, \mathcal{O}_X \rangle$ where $\mathcal{A} = \mathcal{O}_X^\perp$

$$HH^d(X) = \bigoplus_{p=0}^{\dim X} H^{d-p}(X, \Lambda^p TX)$$

$$HH^d(\mathcal{O}_X^\perp) = \bigoplus_{p=0}^{\dim X - 1} H^{d-p}(X, \Lambda^p TX)$$

apply $\Delta^!$:

$$\mathcal{O}_X \boxtimes \mathcal{O}_X \to \Delta_* \mathcal{O}_X \to P$$

$$\omega_X^{-1}[\dim X] \to \bigoplus_{p=0}^{\dim X} \Lambda^p T_X[-p] \to \Delta^! P, \quad HH^d(\mathcal{A}) = H^\bullet(X, \Delta^! P)$$

2) Assume $\langle E, \mathcal{O}_X \rangle$ exc. pair in $D^b(X)$, $\mathcal{A} = \langle E, \mathcal{O}_X \rangle^\perp$: then
vect bundle

$$\cdots \to \bigoplus_{p=0}^{\dim - 1} H^{d-p}(X, \Lambda^p TX) \to HH^d(\mathcal{A}) \to H^{d-\dim+2}(X, E^\perp \otimes E \otimes \omega^{-1})$$

where $E^\perp = \ker(H^\bullet(E) \to E)$

$$\cdots \to \bigoplus_{p=0}^{\dim - 2} H^{d-p}(X, \Lambda^p TX) \to HH^d(\mathcal{A}) \to H^{d-\dim+2}(X, N^\vee \otimes \omega_X^{-1}) \xrightarrow{\star}$$

where $N^\vee = \ker(E^\perp \otimes E \to \Omega_X)$.

If E = line bundle then connecting map $\star$ is zero.

3) $f: X \to Y$ conic bundle $\Rightarrow D^b(X) = \langle \mathcal{A}_X, f^* D^b(Y) \rangle$

$D \underset{i}{\hookrightarrow} Y$ the degeneracy locus of f

$\tilde{D} \xrightarrow[2:1]{} D$ unramified ($\tilde{D} \sim$ moduli of lines in fibers of f).

$\Rightarrow M \in \mathrm{Pic}^0 D$, $M^2 \cong \mathcal{O}_X$. 2-torsion bundle assoc. to covering.

$$HH_d(\mathcal{A}_X) = HH_d(Y) \oplus \bigoplus_{p>0}^{\dim X - 2} H^{p+d}(D, \Omega^p_d \otimes M)$$

$$HH^d(\mathcal{A}_X) = \bigoplus_{p=0}^{\dim Y} H^{d-p}(Y, \ker(\Lambda^p T_Y \to i_*(N \otimes \Lambda^{p-1} T_D)))$$

Nonvanishing conjecture:

> X smooth proj. var., $\mathcal{A} \subset D^b(X)$ admissible : if $HH_\bullet(\mathcal{A}) = 0$ then $\mathcal{A} = 0$.

Rmk: if $\mathcal{A}$ is a CY subcat. then conj holds.

Indeed, $\mathcal{A}$ CY $\Rightarrow HH^\bullet(\mathcal{A}) = HH_\bullet(\mathcal{A})$ up to a shift

Corollary 1: (generation criterion)

> If $\mathcal{A}_1 \dots \mathcal{A}_m$ semiorthogonal in $D^b(X)$, and $HH_\bullet(X) = \oplus HH_\bullet(\mathcal{A}_i)$ then $D^b(X) = \langle \mathcal{A}_1 \dots \mathcal{A}_m \rangle$ semi $\perp$ decomp.

Corollary 2: (Noetherian property)

> Any increasing chain of admissible subcats. $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \dots$ in $D^b(X)$ stabilizes at finite place