T. Pantev – 21/1/08 – Generalized Hodge structures and mirror symmetry

(w/ L. Katzarkov & M. Kontsevich)

$\mathbb{A}^1_{\mathbb{C}}$ affine line, w/ coordinate u

Def: a <u>noncomm. Hodge structure of exp^l type</u> is a triple $(H, \mathcal{E}_B, \underline{iso})$

where
- $H \to \mathbb{A}^1_{\mathbb{C}}$ algebraic $\mathbb{Z}/2$-graded vect. bundle
- $\mathcal{E}_B \to \mathbb{A}^1_{\mathbb{C}} - \{0\}$ local system of $\mathbb{Z}/2$-graded $\mathbb{Q}$-vect. spaces
- $\underline{iso}$: $\mathcal{E}_B \otimes \mathcal{O}_{\mathbb{A}^1 - \{0\}} \xrightarrow{\sim} H_{|\mathbb{A}^1 - \{0\}}$

(→ induces a flat holom. connection ∇ on $H_{|\mathbb{A}^1-\{0\}}$)

These data satisfy

(1) • (nc filtration axiom)^{exp} : $\exists$ holom. frame of H near 0

s.t. $\nabla = d + \sum_{k \geq -2} A_k u^k du$

$A_k \in \mathrm{Mat}_{r \times r}(\mathbb{C})$

& $\exists$ holom. frame of H near ∞ st.

∇ has a logarithmic sing. at ∞

• also, $\exists$ formal isom. $(\mathcal{H}[u^{-1}], \nabla) \cong \bigoplus_{i=1}^{m} \mathcal{E}^{c_i/u} \otimes (R_i, \nabla_i)$

as Laurent series, but not necess. convergent

germ of H at 0

where (R_i, ∇_i) regular at 0

$\mathcal{E}^{c_i/u} = (\mathbb{C}\{u\}, d - d(c_i/u))$

$c_1 \dots c_m$ = the distinct eigenvals. of A_{-2}.

<u>NB:</u> in general Turrittin–Levelt says such an isom. always exists after a cyclic group action base change ($u \leftrightarrow u^n$), ie. working with Puiseux series. "Exponential type" means the base change isn't needed and we can use Laurent series.

(2) ($\mathbb{Q}$-structure axiom)^{exp}: The local system $(H, \nabla)_{|\mathbb{A}^1 - \{0\}}$ induces a local system $\mathcal{S}$ on $S^1 = \mathbb{C}^*/\mathbb{R}_{>0}$

and iso induces a rational structure $\mathcal{S}_B \subset \mathcal{S}$;
We require $\mathcal{S}_B$ to be compatible with Stokes data.

- $\mathcal{S}$ is equipped with Deligne-Malgrange-Stokes filtration labelled by $\lambda \in \mathbb{R}$, by subsheaves:

$$\forall \varphi \in S^1, \ (\mathcal{S}_{\leq \lambda})_\varphi = \left\{ s \;\middle|\; s \in \Gamma(\mathbb{R}_+ e^{i\varphi}, H)^\nabla, \ \|s(re^{i\varphi})\| = O\left(\exp\left(\frac{\lambda + o(1)}{r}\right)\right) \text{ as } r \to 0 \right\}$$

We require this filtration to be rational, ie.

$$\left((\mathcal{S}_{\leq\lambda}) \cap \mathcal{S}_B\right) \otimes \mathbb{C} = \mathcal{S}_{\leq\lambda} \quad \forall \lambda.$$

NB: Deligne-Malgrange thm: $(\mathcal{S}, \mathcal{S}_{\leq\lambda})$ uniquely reconstructs $(\mathcal{H}, \nabla)$ as a meromorphic connection at 0.

(3) (opposedness axiom)$^{\exp}$:
The $\mathbb{Q}$-structure on $\mathcal{S}$ induces a real structure
ie. $\tau : \mathcal{S} \to \mathcal{S}$ antilinear involution.
Let $\hat{H}$ be the holom. bundle on $\mathbb{P}^1$ glued out of $H|_{\{u \leq 1\}}$ and $\gamma^*\left(\overline{H|_{\{|u|\leq 1\}}}\right)$ via τ

$$\left(\gamma : \begin{array}{l} \mathbb{P}^1 \to \mathbb{P}^1 \\ u \mapsto 1/\bar{u} \end{array}\right)$$

Then we require that $\hat{H}$ be trivializable, $\hat{H} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus r}$

- Parts of this have appeared before
  - "semi-∞ H.S." in Mirr. symm.
  - Hertling's TER structures
  - ....

Comm. H.S. V $\mathbb{C}$-vector space, $F^\bullet$ decreasing filtration, $V_\mathbb{Q} \subset V$ $\mathbb{Q}$-subspace
$\mathbb{Q}$ str. axiom: $V_\mathbb{Q} \otimes \mathbb{C} = V$; opposedness axiom: $\overline{F^i}$ .....
pure Hodge str. of weight w

Starting with $(V, F^\bullet, V_{\mathbb{Q}})$, can construct $(H, \mathcal{E}_B, \underline{iso})$ by

- $H \to \mathbb{A}^1_{\mathbb{C}}$ bundle corresp. to $\xi(V, F^\bullet) = \sum u^{-p} F^p V \subset \mathbb{C}[u] \otimes V$
- $H \subset \mathcal{T}_{\frac{w}{2}} \otimes V$, $\mathcal{T}_{\frac{w}{2}} = \left(\mathcal{O}, d - \frac{w}{2}\frac{du}{u}\right)$

  and H is a lattice for the natural connection $\left(d - \frac{w}{2}\frac{du}{u}\right) \otimes id_V$

  This connection has monodromy $(-1)^w \, \mathrm{Id}$

  $\Rightarrow$ preserves any rational structure, e.g. $V_{\mathbb{Q}}$

  $\Rightarrow$ get $\mathcal{E}_B$ + $\underline{iso}$.

<u>Note</u>: $(H, \mathcal{E}_B, \underline{iso})$ is s.t. (H, τ) has a regular singularity.

☆ This gives a functor $(\mathbb{Q}\text{-}HS) \longrightarrow (\mathbb{Q}\text{-}ncHS)^{exp}$
(NB: these are abelian categories!)

factors through quotient by Tate twist $(\cdot) \otimes \mathbb{Q}(1)$

$$\Rightarrow \quad N : (\mathbb{Q}\text{-}HS) / (\cdot) \otimes \mathbb{Q}(1) \longrightarrow (\mathbb{Q}\text{-}ncHS)^{exp}.$$

<u>Lemma</u>: $(\mathbb{Q}\text{-}ncHS)^{exp}$ is a $\mathbb{Q}$-linear abelian category.
The full subcat. of polarizable nc-HS is semisimple.

<u>Lemma</u>: N is fully faithful with essential image = all ncHS with regular sings. and with monodromy id on H^0 and $(-id)$ on H^1.

<u>Main conj</u>: (Kontsevich Soibelman) If C is a smooth compact $\mathbb{Z}/2$-graded category, then $HP_\bullet(C)$ carries a functorial $(ncHS)^{exp}$ (regular).
And if C is graded then this is an ordinary HS.

periodic cyclic homology. (involves parameter u!).

Dual description:

$$(H, \mathcal{E}, \underline{iso}) \begin{array}{l} \rightsquigarrow (H, \nabla) \quad \text{De Rham part} \\ \rightsquigarrow \mathcal{E}_B \quad \text{Betti part} \end{array}$$

can be studied separately, and we can rewrite the def$^{\underline{n}}$ (and in particular gluing) without appealing to Stokes data

Thm: There is an equivalence

$$\begin{pmatrix} (H, \mathcal{E}_B, \underline{iso}) \text{ satisfying} \\ (\text{nc filtration})^{exp} \\ (\mathbb{Q}\text{-structure})^{exp} \end{pmatrix} \Longleftrightarrow \begin{pmatrix} ((H,\nabla), \mathcal{F}_B, f) \text{ s.t.} \\ \bullet\ (H,\nabla) \text{ satisfies } (\text{nc-filtration})^{exp} \\ \bullet\ \mathcal{F}_B \in \mathrm{Constr}(\mathbb{A}^1, \mathbb{Q}), \\ \quad R\Gamma(\mathbb{A}^1, \mathcal{F}_B) = 0 \\ \bullet\ f : \mathcal{F}_B \otimes \mathbb{C} \xrightarrow{\sim} \mathrm{DR}\left(\widehat{i_*\left((H,\nabla)_{|\mathbb{A}^1-0}\right)}\right) \end{pmatrix}$$

$$i : \mathbb{A}^1 - \{0\} \longrightarrow \mathbb{P}^1 - \{0\}$$
$$u \longmapsto u^{-1}$$

$i_* =$ D-module pushforward

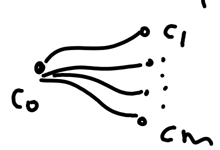
$(\widehat{\cdot}) =$ Fourier transform of D-modules

Thm: If $(H, \mathcal{E}_B, iso)$ is a ncHS of exp. type then it is equiv$^{\underline{t}}$ to

(regular type) : $S = \{c_1 \dots c_m\} \subset \mathbb{A}^1_{\mathbb{C}}$

and a collection $(\mathcal{R}_i, \mathcal{E}_{B,i}, \underline{iso}_i)$ of ncHS with regular sings. at 0.

+ (gluing data): base point $c_0 \in \mathbb{A}^1_{\mathbb{C}}$ and a collection of

paths 

+ linear maps $T_{ij} : (\mathcal{E}_{B,i})_\infty \to (\mathcal{E}_{B,j})_\infty$ for $i \neq j$.

($T_{ij} \sim$ Fourier transf. of Stokes matrices)