S. Gukov - 24/1/08 - Gauge theory and categorification II

- $N=4$ susy Yang-Mills with target $C \times \mathbb{R}^2$ (C: Riem. surface) $\rightsquigarrow$ Hitchin eqns
$$\begin{cases} F_A - \phi\wedge\phi = 0 \\ d_A \phi = 0 \\ d_A * \phi = 0 \end{cases}$$
where A conn. on G-bundle, $\phi \in \Omega^1(C, \mathrm{ad}\, G)$

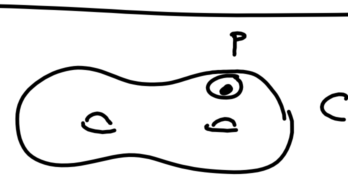
- <u>Hitchin moduli space</u>:
$$\mathcal{M}_H(G,C) = \left\{ (A,\phi) \;\middle|\; \begin{array}{l} F_A - \phi\wedge\phi = 0 \\ d_A\phi = 0 \\ d_A*\phi = 0 \end{array} \right\} \Big/ \mathcal{G}$$
$$= \left\{ \mathcal{A} = A + i\phi \;\middle|\; \begin{array}{l} F_A - \phi\wedge\phi = 0 \\ d_A\phi = 0 \end{array} \right\} \Big/ \mathcal{G}_{\mathbb{C}}$$
$$= \mathcal{M}^{G_{\mathbb{C}}}_{flat}(C). \quad \text{flat } G_{\mathbb{C}}\text{-connections}$$

$\mathcal{M}_H(G,C)$ is <u>hyperkähler</u>: $I, J, K = IJ$; $\omega_I, \omega_J, \omega_K$

(J is the "natural" $\mathbb{C}$ str. in above description).

Thus we have flexibility in choosing to consider either A or B model on $\mathcal{M}_H(G,C)$.

- <u>Surface operators</u>:
$$\begin{array}{ccc} M & = & \mathbb{R}^2 \times C \\ \cup & & \cup \\ D & = & \mathbb{R}^2 \times p \end{array} \quad \Rightarrow$$



ie. consider a <u>punctured</u> Riem. surface (C,p), and allow (A,ϕ) to have a <u>singularity</u> at p.

Local model: on $D' =$ punctured disc w/ coords $z = re^{i\theta}$ $(p: z=0)$.
$$\begin{cases} A = \alpha \dfrac{r\,d\theta}{r} + \cdots \\ \phi = \beta \dfrac{dr}{r} - \gamma\, d\theta + \cdots \end{cases}$$
$\alpha, \beta, \gamma \in \mathfrak{t} = \mathrm{Lie}(\mathbb{T})$, $\mathbb{T} \subset G$ max. torus

Gauge equivalence: $u \in \Lambda_{\text{cochain}}(G)$, $f = e^{u\theta}$

acts by $\alpha \mapsto \alpha + u$

Hence: $(\alpha, \beta, \gamma) \in (\mathbb{T} \times \mathfrak{t} \times \mathfrak{t})/W$

- supersymmetry $N=4$ (parameters come in 4-tuples)
  [e.g in $N=2$ A-model, introduce B-field to complexify ω].

$\Rightarrow$ need an extra "quantum" param. $\eta \in {}^L\mathbb{T}$ max. torus of <u>Langlands dual</u> of G.

$$\left[\text{on } D \simeq \mathbb{R}^2, \text{ add phase factor } \exp\left(i\eta \int_D F_A\right) \right]$$

$$\Rightarrow (\alpha, \beta, \gamma, \eta) \in (\mathbb{T} \times \mathfrak{t} \times \mathfrak{t} \times {}^L\mathbb{T})/W.$$

$\mathcal{M}_H(G, C, p; \alpha, \beta, \gamma, \eta)$ $\dim H^2(\mathcal{M}_H(\dots)) = \operatorname{rank} G$.

Model	Complex moduli	Kähler moduli
I	$\beta + i\gamma$	$\alpha + i\eta$
J	$\gamma + i\alpha$	$\beta + i\eta$
K	$\alpha + i\beta$	$\gamma + i\eta$

<u>NB:</u> B-model for J depends on $\gamma + i\alpha$ but <u>not</u> on $\mathbb{C}$ str. of C (it appears in A-model for J)

However, e.g. B-model for I depends partially on $\mathbb{C}$ str. of C.

<u>Ex:</u> $G = SU(2)$, $C = \mathbb{C} \setminus \{0\}_p$

can see e.g. by thinking of flat $G_{\mathbb{C}}$ connections

$\Rightarrow \mathcal{M}_H \cong \mathbb{H}^2 /\!/\!/ U(1)$ hk quotient at $\vec{\mu} = (\mu_I, \mu_J, \mu_K) = (\alpha, \beta, \gamma)$

for $\alpha = \beta = \gamma = 0$, get $\mathcal{M}_H \simeq \mathbb{C}^2/\mathbb{Z}_2$; otherwise $\mathcal{M}_H = T^*\mathbb{CP}^1$, with $\eta = \int_{\mathbb{CP}^1} B$ (B-field for <u>any</u> of I, J, K)

Ex: (Painlevé VI)

$$G = SU(2), \qquad C = \mathbb{CP}^1 \setminus \{p_1, p_2, p_3, p_4\}$$

$$\mathcal{M}_H(G, C, \dots)_J = \{ \rho : \pi_1(C) \to G_{\mathbb{C}} \,/\, \rho(\gamma_i) \in \mathcal{C}_i \} /\sim$$

conj. classes at the 4 punctures: in terms of params. $V_i = \exp(2\pi i(\gamma_i + i\alpha_i))$

$$= \{ U_i \,/\, U_i \in \mathcal{C}_i, \; U_1 U_2 U_3 U_4 = Id \} /\sim .$$

Can write explicitly in terms of

variables $\begin{cases} x_1 = tr(U_3 U_2) \in \mathbb{C} \\ x_2 = tr(U_1 U_3) \\ x_3 = tr(U_2 U_1) \end{cases}$ & params.
$\theta_1 = a_1 a_4 + a_2 a_3$
$\theta_2 = - - - -$
$\theta_3 = - - - - .$
$\theta_4 = a_1 a_2 a_3 a_4 + \sum_1^4 a_i^2 - 4$

where $a_i = tr(U_i)$ $i = 1..4$ are determined by parameters α, γ.

Then get $\mathcal{M}_H(\dots)_J = \{ (x_1, x_2, x_3) \in \mathbb{C}^3 / f(x_i, \theta_m) = 0 \}$

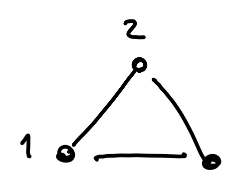
where $f(x_i, \theta_m) = x_1 x_2 x_3 + \sum_1^3 (x_i^2 - \theta_i x_i) + \theta_4$

$\Rightarrow$ cubic surface in $\mathbb{C}^3$ w/ eqn depending on parameters.

- $Br_3 \ni \sigma_i$ $i = 1, 2, 3$ acts on $\mathcal{M}_H(\dots)_J$:

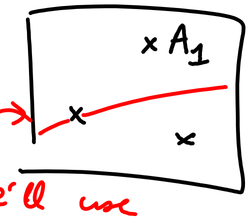
σ_i: $x_i \mapsto \theta_j - x_j - x_k x_i$ $\{i, j, k\} = \{1, 2, 3\}$ $\theta_i \mapsto \theta_j$
$x_j \mapsto x_i$ $\theta_j \mapsto \theta_i$
$x_k \mapsto x_k$ θ_k, θ_4 stay

($\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$
$\sigma_k = \sigma_i \sigma_j \sigma_i^{-1}$



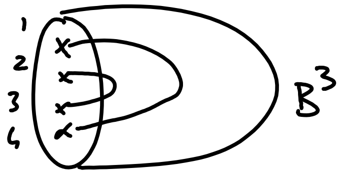
(place 4 at ∞))

- Take $a_i = a$ all equal: then $\mathcal{M}_K$ has 3 A_1 sings.



$\widetilde{\mathbb{B}}$ brane that we'll use to define invts

We associate $\widetilde{\mathbb{B}}$ to

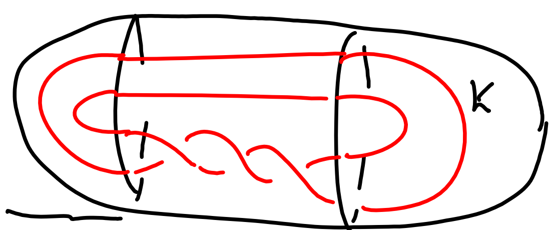


i.e. $\widetilde{\mathbb{B}} = \{ U_1 = U_4^{-1},\ U_2 = U_3^{-1} \}$

$= \{ x_1 = tr(U_3 U_2) = 2 \}$

$\widetilde{\mathbb{B}}$ passes through one of the A_1-singularities, but we define Hom's ignoring the sing. pt.

Then



$\widetilde{\mathbb{B}}$

$\beta \in Br_3$

$\Rightarrow \phi_\beta \in Aut(\mathcal{M}_K)$

$\phi_\beta(\widetilde{\mathbb{B}})$

$\Rightarrow H_K := \mathrm{Hom}(\widetilde{\mathbb{B}}, \phi_\beta(\widetilde{\mathbb{B}}))$

<u>Ex</u>: $T_{2,k}$ torus knot: $\beta = \sigma_1^k$

$\Rightarrow \dim H_{T_{2,k}} = 2\sigma(T_{2,k})$