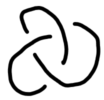
S. Gukov – 22/1/08 – Gauge theory and categorification

Categorification := promote invariants from

Number $\rightleftarrows$ Vector space $\rightleftarrows$ Category $\rightleftarrows$ 2-category ....

(dim. ; K-group)

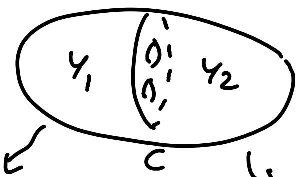
<u>Ex:</u>  $\to$ Jones polynomial $J(q)$

categorifies to Khovanov homology $Kh(K)$, $\chi_q(Kh) = J(q)$.

- 3D TQFT is a functor 3-mfld $Y \rightsquigarrow$ number $Z(Y)$

  surface $C \rightsquigarrow$ vector space $\mathcal{H}_C$

  Heegaard decomp$^{\underline{n}}$. $Y = Y_1 \cup_C Y_2$



$\langle Y_1 | \in \mathcal{H}_C^*$ $\qquad$ $|Y_2\rangle \in \mathcal{H}_C$

$\Rightarrow Z(Y) = \langle Y_1 | Y_2 \rangle$

- 4D TQFT: 4-mfld $M \rightsquigarrow$ number $Z(M)$

  gauge theory on $\mathbb{R}\times Y^3 \rightsquigarrow$ vector space $\mathcal{H}_Y$

  gauge theory on $\mathbb{R}^2 \times C \rightsquigarrow$ category $\mathcal{F}(C)$

[NDDA: Ozsvath-Szabo: 4-mfld inv't $= HF(Y)$ $= \mathrm{Fuk}(\mathrm{Sym}^g C)$]

More precisely, in gauge theory:

- M 4-mfld, with data $G \rightsquigarrow \mathcal{M}(G,M)$ moduli space of sol$^{\underline{ns}}$

  $\rightsquigarrow Z(M) = \chi(\mathcal{M}(G,M))$ numerical invariant

- Y 3-mfld, gauge theory on $\mathbb{R}\times Y \Rightarrow \mathcal{H}_Y = H^*(\mathcal{M}(G,Y))$

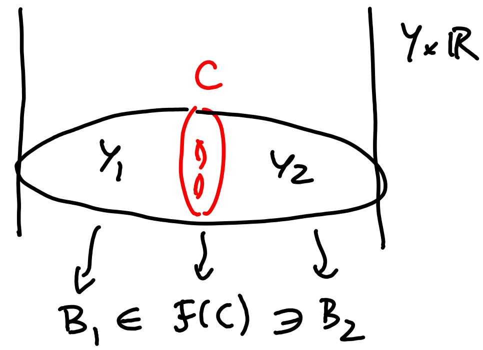
- C 2-mfld, gauge theory on $\mathbb{R}^2 \times C \Rightarrow \mathcal{F}(C) = \begin{cases} D^b_{Ch}(\mathcal{M}(G,C)) & \text{B-model} \\ \mathrm{Fuk}(\mathcal{M}(G,C)) & \text{A-model} \end{cases}$

  reduces to <u>σ-model</u> on $\mathbb{R}^2$ w/ <u>target $\mathcal{M}(G,C)$</u>

[why the category? think of it as needed for 4D theory on mfld w/ <u>corners</u> and relating together the various boundary conditions at corners].

Going back to a Heegaard decomposition: in a 4D TFT, if

Y 3-mfld is $Y_1 \cup_C Y_2$



$C \longmapsto \mathcal{F}(C)$ category

$Y_1, Y_2 \mapsto$ <u>objects</u> $B_1, B_2 \in \mathcal{F}(C)$

("branes")

Then $\mathcal{H}_Y$ = vector space $\mathrm{Hom}_{\mathcal{F}(C)}(B_1, B_2) = \begin{cases} HF_*^{symp}(B_1, B_2) & \text{A-model} \\ \mathrm{Ext}^*(B_1, B_2) & \text{B-model} \end{cases}$

- <u>Ex</u>: Donaldson-Witten gauge theory:

  $\mathcal{H}_Y = HF_*^{inst}(Y)$

  $\mathcal{M}(G, C) = \mathcal{M}^G_{flat}(C)$ moduli of flat G-connection over C

  The above vision suggests: $Y_i \rightsquigarrow B_i \subset \mathcal{M}(G, C)$

  flat G-connections which extend to Y_i

  $B_i \subset \mathcal{M}(G, C)$ is a Lagrangian submfld.

  $HF_*^{inst}(Y) = HF_*^{symp}(B_1, B_2)$ <u>Atiyah-Floer conjecture</u>

- <u>Ex</u>: Seiberg-Witten: moduli space of sol^{ns} to <u>vortex eqns</u> on C.

  4D SW: $\begin{cases} F_A^+ + i(\psi\bar{\psi})_+ = 0 \\ \not{D}_A \psi = 0 \end{cases}$ $\rightsquigarrow$ 3D SW monopole homology $\rightsquigarrow$ 2D: vortex eqns

  Since $\mathcal{M}$. for vortex eqns $\leftrightsquigarrow$ symmetric product,

  this motivates Ozsvath-Szabo theory in our context.

Surface operators

(w/ Witten)

|| Operators in 4D gauge theory supported on 2D surfaces $D \subset M$.

(cf. point operator: $\mathcal{O}(p) = \mathrm{Tr}(\varphi^2)$)

- Closed $D^2 \subset M^4 \rightsquigarrow Z(D, M)$ inv't of pair (M, D) cf. Kronheimer-Mrowka

- $$\begin{matrix} M = \mathbb{R} \times Y \\ \cup \qquad \cup \\ D = \mathbb{R} \times K \end{matrix} \rightsquigarrow \mathcal{H}_{Y; K}$$

- $$\begin{matrix} M = \mathbb{R}^2 \times C \\ \cup \qquad \Cup \\ D = \mathbb{R}^2 \times p \end{matrix} \rightsquigarrow \mathcal{F}(C, p)$$

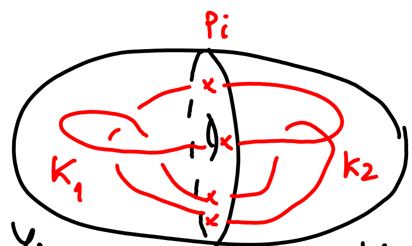
<u>Note:</u> the mapping class group of (C, p_i) acts on $\mathcal{F}(C, p_i)$

<u>Ex:</u> $C = \mathbb{C} \setminus \{p_1 \cdots p_n\}$

$\Rightarrow Br_n = \pi_1(\mathrm{Conf}^n(\mathbb{C}))$ acts on $\mathcal{F}(C, p_i)$.

$\beta \in Br_n \longmapsto \phi_\beta$, $\quad \phi_\beta : \mathcal{F}(C) \to \mathcal{F}(C)$

<u>Example:</u>



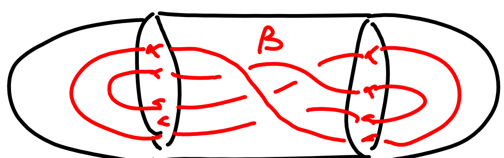
$Y_1 \qquad C \qquad Y_2$

$\Downarrow \qquad \Downarrow \qquad \Downarrow$

$B_1 \in \mathcal{F}(C, p_i) \ni B_2$

$$\begin{matrix} K = K_1 \cup_{\{p_i\}} K_2 \\ \cap \quad \cap \qquad \cap \\ Y = Y_1 \cup_C Y_2 \end{matrix}$$

Braid group action on $\mathcal{F}(C, p_i)$ allows us to view inv't of (Y, K) as follows:
e.g. $Y = S^3$



simplest config. $\Downarrow$ "simple" brane $\widetilde{B} \in \mathcal{F}(C, p_i)$

C braid $\beta \in Br_n$ $\Downarrow$ C

same simple config $\Downarrow$ same $\widetilde{B} \in \mathcal{F}(C, p_i)$

$\phi_\beta : \mathcal{F}(C) \to \mathcal{F}(C)$

Then the 3-mfd relative inv't is $\mathcal{H}_{Y, K} = \mathrm{Hom}_{\mathcal{F}(C)}(\phi_\beta(\widetilde{B}), \widetilde{B})$

Cf. eg. Seidel-Smith or Cautis-Kamnitzer versions of Khovanov homology, $\widehat{HFK}$, ...