M. Kontsevich – 24/1/08 – Motivic DT invts III

$\mathcal{C}$ A_∞ 3-CY category$/k$ ie. $\sum_i \operatorname{rk} \operatorname{Ext}^i(\mathcal{E},\mathcal{F}) < \infty$, $\operatorname{Ext}^i(\mathcal{E},\mathcal{F})^* \cong \operatorname{Ext}^{3-i}(\mathcal{F},\mathcal{E})$;

- char $k = 0$, cyclic A_∞-alg. category $\langle m_n(f_1, \dots f_n), f_{n+1} \rangle$ is $\mathbb{Z}/_{n+1}$-symmetric
- char $k > 0$, use Kontsevich–Soibelman def^n.

* $\forall \mathcal{E}$, "minimal model": formal power series on $\operatorname{Ext}^1(\mathcal{E},\mathcal{E})$

$$\psi_{\mathcal{E}}(\alpha) := \sum_{n \geq 2} \frac{\langle m_n(\alpha, \dots, \alpha), \alpha \rangle}{n+1} \qquad \text{for } \alpha \in \operatorname{Ext}^1(\mathcal{E},\mathcal{E}).$$

(NB: $\psi_{\mathcal{E}} = O(\alpha^3)$)

<u>NB</u>: critical points of $\psi_{\mathcal{E}}$ near 0 are honest deform$^{\text{s}}$ of $\mathcal{E}$ (Maurer-Cartan sol$^{\text{ns}}$)

* $$\boxed{\underset{\text{motivic weight of } \mathcal{E}}{W(\mathcal{E})} := \left(\mathbb{L}^{1/2}\right)^{\sum_{i \leq 0} (-1)^i \operatorname{rk} \operatorname{Ext}^i(\mathcal{E},\mathcal{E})} \left(1 - \left[H^*(\text{Milnor fiber of } \psi_{\mathcal{E}})\right]\right)}$$

where $\mathbb{L} = [H_c^*(\mathbb{A}^1)]$

	Milnor fiber	Thom-Sebastiani thm
1	$X \ni x_0$, $f \in \mathcal{O}(X)$, $f(x_0) = 0$ $\mathbb{C}$ analytic, smooth $\rightsquigarrow$ bundle of cohomologies over the punctured disc $\{0 < \lvert z \rvert < \varepsilon\}$ $H^*(\underbrace{f^{-1}(z)}_{\text{milnor fiber } MF(f)} \cap \{\lvert x - x_0 \rvert < \varepsilon^{1/d}\}, \mathbb{Z}/_{\gamma})$	$f = g \boxplus h$ on $X = \underset{g}{X_1} \times \underset{h}{X_2}$ $(1 - \chi(MF(f))) = (1 - \chi(MF(g)))(1 - \chi(MF(h)))$
2	H^* mixed Hodge structures $\rightsquigarrow$ variations of mixed HS $/\mathbb{C}^*$	$\rightarrow$ Hodge polynomial in 2 variables $\sum a_{R_1,R_2} z_1^{R_1} z_2^{R_2}$ $R_1, R_2 \in \mathbb{Q}$, $R_1 + R_2 \in \mathbb{Z}$. Also have product property.

③ Denef-Loser: defined a formal Milnor fiber (formal combination of varieties w/ action of roots of unity)

Def: uses resolution of sings, hence also works for formal power series.

④ SGA: fix $\ell \neq \mathrm{char}\, k > 0$

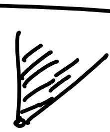
Get "nearby cycles" in K_0(ℓ-adic rep$^{\text{ns}}$ of $\mathrm{Gal}(k((t)))$).

← can extend every ℓ-adic sheaf to $\mathbb{G}_m$ tam at ∞, apply Fourier transform & get an elt of $\mathbb{Q}_\ell(\sqrt[p]{1})$

$\star$ This also works with parameters. (for a family of functions).

Ex: $\varphi = x^3 \rightsquigarrow$ Hodge polynomial $z_1^{1/3} z_2^{2/3} + z_1^{2/3} z_2^{1/3}$

Informal claim: Over $\mathbb{F}_q$:

given a sector V , $V = \mathbb{R}_+ \exp(iI)$, I interval of length $< \pi$

Recall we have an element A_V associated to it: we claim that

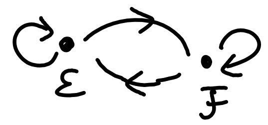
$$(\star) \quad A_V = \sum_{\substack{\mathcal{E} \in \mathcal{C}_I \\ \uparrow \\ \text{part of category with phases} \in I}} \frac{1}{\# \mathrm{Aut}\, \mathcal{E}} \underbrace{\left(q^{1/2}\right)^{\sum_{i \leq 0} (-1)^i \mathrm{rk}\, \mathrm{Ext}^i(\mathcal{E},\mathcal{E})} \Big(1 - \mathrm{Tr}\, \overset{\substack{\text{Gal-invt part?} \\ \swarrow}}{\mathrm{Fr}} \big(\mathrm{MF}(\varphi_{\mathcal{E}})\big)\Big)}_{\substack{\text{this coefficient} \\ \text{comes from the} \\ \text{above derivation.}}} \cdot \underset{\substack{\uparrow \\ [\mathcal{E}] \in \Lambda \text{ is} \\ \mathrm{cl}(\mathcal{E})}}{e_{[\mathcal{E}]}}$$

$\rightsquigarrow A_V$ is defined as an element of the quantum torus of Λ.

[this is the part where we go from motivic Hall algebra to quantum torus]

- Basic identity: Over $\mathbb{F}_q$:

$$\forall \mathcal{E}, \mathcal{F}, \sum_{\alpha \in \mathrm{Ext}^1(\mathcal{F},\mathcal{E})} W(\mathcal{E} \oplus_\alpha \mathcal{F}) = q^{?}\, W(\mathcal{E})\, W(\mathcal{F})$$

Given $\mathcal{E}$ & $\mathcal{F}$: 

$$\mathrm{Ext}^1(\mathcal{E}\oplus\mathcal{F}, \mathcal{E}\oplus\mathcal{F}) = V_1 \oplus V_2 \oplus V_3$$
$$V_1 = \mathrm{Ext}^1(\mathcal{F},\mathcal{E})$$
$$V_2 = \mathrm{Ext}^1(\mathcal{E},\mathcal{F})$$
$$V_3 = \mathrm{Ext}^1(\mathcal{E},\mathcal{E}) \oplus \mathrm{Ext}^1(\mathcal{F},\mathcal{F}).$$

Then $\psi_{\mathcal{E}\oplus\mathcal{F}}$: series at $0 \in V_1 \oplus V_2 \oplus V_3$
deg. $+1, -1, 0$

Thom-Sebastiani thm $\Rightarrow$

$$\sum_{\alpha\in V_1} \left(1 - H^\bullet(MF\, \psi|_{(\alpha,0,0)})\right) = q^{\mathrm{rk}\, V_1}\left(1 - H^\bullet(MF\,\tilde{\psi})\right)$$

Milnor fiber in 3 variables,
but for Taylor exp^n of ψ at $(\alpha,0,0)$, not at origin

$\tilde{\psi} = \psi|_{(0,0,V_3)}$
function on V_3 only.

$\Rightarrow$ gives the above identity.

- Think now over $\mathbb{C}$: ψ on $V_1 \oplus V_2 \oplus V_3$:

$$LHS := H^\bullet_c\left(\left\{(z_1,z_2,z_3) \,\middle|\, \begin{array}{l}\psi(z_1,z_2,z_3)=t,\ |t|=\varepsilon \\ |z_1|\le \varepsilon^{-1/10},\ |z_2|,|z_3|\le\varepsilon^{1/100}\end{array}\right\}\right)$$

$$RHS := H^\bullet_c\left(\{(0,0,z_3) \mid \psi(0,0,z_3)=t,\ |t|=\varepsilon,\ |z_3|\le\varepsilon^{1/100}\}\right)\cdot H^\bullet_c\left(\{(z_1,0,0),\ |z_1|<1\}\right)$$

Rescaling $z_1 \mapsto z_1\lambda$
$z_2 \mapsto z_2/\lambda$, $\lambda\in\mathbb{C}^*$ $\rightsquigarrow$ LHS $\ni$ pieces $|z_1||z_2|>0$
and $|z_1|,|z_2| = 0$.

$\bigstar$

Corollary: We obtain a map for C cat./$\mathbb{C}$,

$$\mathrm{Stab}(C) \xrightarrow[(\bigstar)]{} \mathrm{Stab}(\text{quantum torus}) \qquad e_{\gamma_1}e_{\gamma_2} = L^{\frac12\langle\gamma_1,\gamma_2\rangle} e_{\gamma_1+\gamma_2}$$
$$\mathcal{D} \rightsquigarrow \mathbb{Q}[z_1,z_2]$$

(ie collection of elements A_V for each angular sector V)

Potential candidates: to apply this to:

1) X noncompact CY 3-fold $\supset Z$ proper, $\quad \mathcal{C} = \mathrm{Perf}_{\mathrm{Supp}\, Z}(X)$
2) A finite dim! A_∞-algebra (CY 3: $A^i \otimes A^{3-i} \to k$), $\quad \mathcal{C} = \mathrm{Perf}(A\text{-mod})$
3) M closed oriented C^∞ 3-manifold $= K(\pi, 1)$, $\quad \mathcal{C} = D_{\text{finite}}(\mathbb{C}[\pi]\text{-modules})$
4) Quivers with potentials.

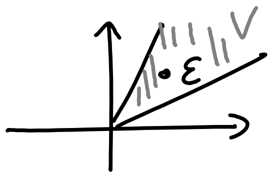
Example 1:

$\mathcal{C} = \langle \mathcal{E} \rangle$, $\quad \mathcal{E}$ with $\mathrm{Ext}^i(\mathcal{E}, \mathcal{E}) = H^\bullet(S^3)$

$(K_0(\mathcal{C}) = \mathbb{Z})$. $\quad$ Stab cond.: • fix $Z(\mathcal{E}) \in \mathbb{H} = \{\mathrm{Im}\, z > 0\}$

• $\mathcal{C}^{ss} = \mathcal{E}, \mathcal{E} \oplus \mathcal{E}, \ldots\ldots$

Then for $V \ni Z(\mathcal{E})$



NB: rank $\mathrm{Ext}^0(\mathcal{E}^{\oplus n}, \mathcal{E}^{\oplus n}) = n^2$

$$\Rightarrow A_V = \sum_{n \geq 0} \frac{q^{n^2/2}}{\# GL(n, \mathbb{F}_q)} x^n \quad \text{where } x = e_{[\mathcal{E}]}$$

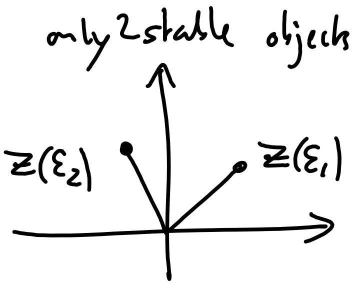
autom. of $\mathcal{E}^{\oplus n} / \mathbb{F}_q$

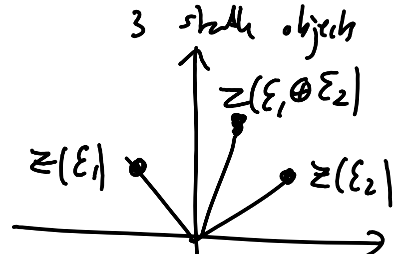
$\# GL(n, \mathbb{F}_q) = (q^n - 1)(q^n - q) \ldots (q^n - q^{n-1})$

"quantum dilogarithm" $\quad \mathbb{L}(x)$

Example 2: $\quad \mathcal{C} = \langle \mathcal{E}_1, \mathcal{E}_2 \rangle \quad \mathrm{Ext}^\bullet(\mathcal{E}_i, \mathcal{E}_i) = H^\bullet(S^3)$

$\mathrm{rk}\, \mathrm{Ext}^1(\mathcal{E}_2, \mathcal{E}_1) = 1$; $\quad \mathrm{Ext}^{\neq 1}(\mathcal{E}_2, \mathcal{E}_1) = 0$





In the quantum torus, $xy = qyx \qquad (x = e_{[\mathcal{E}_1]}, y = e_{[\mathcal{E}_2]})$

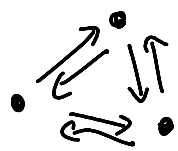
$\mathbb{L}(x)\mathbb{L}(y) = \mathbb{L}(y)\mathbb{L}(xy)\mathbb{L}(x) \quad \Rightarrow A_V$'s invt under wall crossing ✓.

- More generally: $(\mathcal{E}_i)_{i\in I}$ collection of spherical objects.

  $\mathrm{rk}\,\mathrm{Ext}^\alpha(\mathcal{E}_i,\mathcal{E}_j)=0$ for $\alpha\neq 1,2$ & $i\neq j$

  i.e. Ext^1's form a quiver  without self-loops

  A_∞-CY-str. $\Longleftrightarrow$ potential ψ

  for A_3 configuration  + generic potential

  $$\rightsquigarrow A_{\text{upper half plane}} = \sum c_{ijk}\, x_1^i x_2^j x_3^k \in \text{quantum torus}$$

  $c_{ijk}\in\mathbb{Q}(q^{1/2})$

  $c_{000}=1$.

  $\exists\, b_{m_1,m_2,n}\in\mathbb{Q}(q^{1/2})$

  $m_1\geq 0$, $|m_2|\leq m_1$, $n\geq 0$, with $b_{000}=1$, s.t.

  $$c_{n_0,n_1,n_2}=\sum_{\ell\geq 0}\frac{q^{\ell(n_2-n_1)}}{[\ell]!}\,b_{n_0,\,n_0-\ell-n_1,\,n_2}$$

  $[\ell]! := (q^\ell-1)(q^{\ell-1}-1)\cdots$

  $$=\sum_{\ell\geq 0}\frac{q^{\ell(n_2-n_0)}}{[\ell]!}\,b_{n_0,\,n_0+\ell-n_1,\,n_2}$$

  can check various identities ....

- In spite of ugly denominators in all these expressions, expect there's a meaningful limit as $\begin{cases} q\to 1. \\ q^{1/2}\to -1. \end{cases}$

  $\rightsquigarrow$ "DT invts"