M. Kontsevich – 23/1/08 – Motivic DT invts II

**Wall-crossing:** $\Lambda \simeq \mathbb{Z}^r$ lattice, $\mathfrak{g} = \bigoplus_{\gamma \in \Lambda} \mathfrak{g}_\gamma$ alg./$\mathbb{Q}$

**Def.** $\mathrm{Stab}(\mathfrak{g}) = \{ (Z, (a_\gamma)) / Z : \Lambda \to \mathbb{C},\ a_\gamma \in \mathfrak{g}_\gamma,$
stability conditions
$\exists$ quadratic form $Q$ on $\Lambda \otimes \mathbb{R}$ s.t.
$Q_{|\ker Z \otimes \mathbb{R}} < 0$, and $a_\gamma \neq 0 \Rightarrow Q(\gamma) > 0 \}$

**Claim:** $\mathrm{Stab}(\mathfrak{g})$ has a natural Hausdorff topology s.t. projection to $Z$, $\mathrm{Stab}(\mathfrak{g}) \to \mathbb{C}^r = \mathrm{Hom}(\Lambda, \mathbb{C})$, is a local homeomorphism

**Categories and stability** (Bridgeland):

$\mathcal{C}$ triangulated cat., $\Lambda \simeq \mathbb{Z}^r$ finite rank lattice,
$cl : K_0(\mathcal{C}) \to \Lambda$

Numerical function on $K_0(\mathcal{C})$: $\varphi : \mathcal{C} \to D^b(\mathrm{Vect}_k)$
induces $K_0(\mathcal{C}) \to K_0(\text{---}) = \mathbb{Z}$

$$\mathrm{Stab}(\mathcal{C}, cl) := \left\{ (Z, \mathcal{C}^{ss}, \mathrm{Arg}) \,\middle/\, \begin{array}{l} Z : \Lambda \to \mathbb{C} \\ \mathcal{C}^{ss} \subset \mathrm{Ob}(\mathcal{C}) \text{ full subcat.} \\ \mathrm{Arg} : \mathcal{C}^{ss} \to \mathbb{R} \end{array} \right\} \text{ s.t.}$$

Axioms: 1 support: $\exists Q$ quadr. form on $\Lambda \otimes \mathbb{R}$ st.
$Q_{|\ker Z} < 0$ and $Q(cl\, \mathcal{E}) > 0 \ \forall \mathcal{E} \in \mathcal{C}^{ss}$

2 $\forall \mathcal{E} \in \mathcal{C}^{ss}$, $Z(cl(\mathcal{E})) \in \mathbb{R}_+ e^{i \mathrm{Arg}(\mathcal{E})}$

3 $\mathcal{C}^{ss}$ is stable under shifts, and $\mathrm{Arg}\, \mathcal{E}[1] = \mathrm{Arg}\, \mathcal{E} + \pi$

4 If $\mathrm{Arg}\, \mathcal{E}_1 > \mathrm{Arg}\, \mathcal{E}_2$, $\mathcal{E}_i \in \mathcal{C}^{ss}$ then $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) = 0$

5 $\forall \mathcal{F} \in \mathcal{C}$, $\exists n \geq 0$, $0 = \mathcal{F}_0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \dots \to \mathcal{F}_n = \mathcal{F}$,
$\mathcal{E}_i = \mathrm{Cone}(\mathcal{F}_i / \mathcal{F}_{i-1}) \in \mathcal{C}^{ss}$, $\mathrm{Arg}\, \mathcal{E}_1 > \mathrm{Arg}\, \mathcal{E}_2 > \dots$

(NB: the support axiom isn't present in Bridgeland but ensures discreteness of wall-crossings under truncation @ energy bound).

Equivalent formulation: Instead of specifying $C^{ss}$ and Arg, we can describe sets of objects whose HN-filtration lies entirely inside a given interval $I$:

$$\mathrm{Stab}(C, cl) = \left\{ \left(Z, \{P(I)\}_{I \text{ interval}}\right) \;\middle|\; \begin{array}{l} \forall I \subset \mathbb{R} \text{ interval} \\ \bullet\ P(I) \subset C \text{ full subcat.} \\ \bullet\ P(I+\pi) = P(I)[1] \\ \bullet\ \text{If } I_1 < I_2 \quad (\text{“to the left of”}) \\ \quad \text{then } \mathrm{Hom}(P(I_2), P(I_1)) = 0 \\ \bullet\ \text{If } I = I_1 \sqcup I_2,\ I_1 < I_2 \text{ adjacent} \\ \qquad (\text{eg. } I_1 = [a,b),\ I_2 = [b,c)) \\ \quad P(I) := \text{collection of all extensions} \\ \qquad P(I_2) \to ? \to P(I_1) \\ \bullet\ P(\mathbb{R}) = C \end{array} \right\}$$

Thm (Bridgeland): $\mathrm{Stab}(C)$ is Hausdorff and $pr_Z : \mathrm{Stab}(C) \to \mathbb{C}^r = \mathrm{Hom}(\Lambda, \mathbb{C})$ is locally homeo

(important point: we fix in advance the lattice $\Lambda$ to protect ourselves from jumps in rank $K_0(C)$ when deforming $C$).

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## Ind-constructible categories

($\to$ relate stability cond$^{ns}$ on categories vs. on Lie algebras).

$C$ triangulated cat. over a finite field $\mathbb{F}_q$.

$\to$ Hall algebra:

$\mathrm{Hall}(C)$ = basis: all isom. classes of objects $[\mathcal{E}]$, $\mathcal{E} \in C$

assoc. alg. $/\mathbb{Q}$

& assoc. product
$$[\mathcal{E}_1] \cdot [\mathcal{E}_2] = q^{-\sum_{i \geq 0} (-1)^i \operatorname{rk} \mathrm{Ext}^i(\mathcal{E}_2, \mathcal{E}_1)} \sum_{\alpha \in \mathrm{Ext}^1(\mathcal{E}_2, \mathcal{E}_1)} (\text{ext}^n \text{ of } \mathcal{E}_1 \text{ by } \mathcal{E}_2 \text{ given by } \alpha)$$

- Given $cl : K_0(C) \to \Lambda$, get decomposition $Hall(C) = \bigoplus_{\gamma \in \Lambda} Hall(C)_\gamma$

  $\Lambda$-graded Lie algebra.

- We have a natural map $Stab(C) \to Stab(Hall(C))$ :

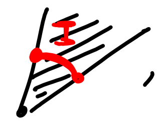
  Recall the two definitions of $stab(g)$:

  ① $stab(g) = \{ (Z, (a_\gamma)_{\gamma \in \Lambda}) \,/\, \dots \}$

  ② $stab(g) = \{ (Z, (A_V)_{V \text{ angular sector}}) \}$

  $$A_V \in g_V \text{ is } \sum_{Z(\gamma) \in V} a_\gamma$$

- Assume we have a stab. cond/$C$ s.t. $\left\{ \mathcal{E} \in C^{ss} \,\middle/\, \begin{array}{l} cl(\mathcal{E}) = \gamma \text{ given} \\ Arg(\mathcal{E}) \text{ given} \end{array} \right\}$ is infinite

  Then for an angular sector $V =$ 

, $V = \mathbb{R}_+ \exp(iI)$ $(\text{length}(I) < \pi)$.

  We set $$A_V := \sum_{\substack{\text{isom. classes of} \\ \mathcal{E} \in \mathcal{P}(I)}} \frac{1}{\# Aut\, \mathcal{E}} [\mathcal{E}] = 1 + \dots\dots$$

- What to do if $C$ is def$^d$ over a base field $k$, not $\mathbb{F}_q$?

  $\Rightarrow$ notion of ind-constructible category :=

$C$ $A_\infty$-category, s.t.: $Ob\ C = \coprod_{\substack{\text{infinite countable} \\ \text{union}}} \text{constr. sets}/k \,\Big/\, \text{Affine alg. group}$

For $i \in \mathbb{Z}$, $\begin{array}{c} Hom^i \\ \downarrow \\ Ob\,C \times Ob\,C \end{array}$ is a constructible sheaf.

$m_1, m_2, \dots$ are algebraic

Finiteness condition $\Rightarrow$ for a stability condition, $C^{ss}_{\gamma, A_n}$ are constructible

- Given $X$ constr. set$/k$, $X \circlearrowleft G$, poor man's motivic functions:

$$\mathrm{Fun}(X) = \mathbb{Z}\text{-module generated by } \begin{matrix} Y \\ \downarrow \\ X \end{matrix} \quad \begin{matrix} Y \text{ constructible set} \\ \text{constr. map} \end{matrix}$$

mod relations $\begin{bmatrix} Y \\ \downarrow \\ X \end{bmatrix} = \begin{bmatrix} Y_1 \\ \downarrow \\ X \end{bmatrix} + \begin{bmatrix} Y_2 \\ \downarrow \\ X \end{bmatrix}$ if $Y = Y_1 \sqcup Y_2$

If $k = \mathbb{F}_q$, $\forall n \geq 1$, $\mathrm{Fun}(X) \to \mathbb{Z}$-valued function on $X(\mathbb{F}_{q^n})$:

$$\begin{bmatrix} Y \\ \downarrow \\ X \end{bmatrix} \longmapsto \left( x \in X(\mathbb{F}_{q^n}) \longmapsto \#\{ y \in Y(\mathbb{F}_{q^n}) / y \mapsto x \} \right)$$

- $C$ ind-constructible $\underbrace{\text{3-dim. CY}}$ $A_\infty$-category

  ↳ i.e. $\mathrm{rk}\ \mathrm{Ext}^i < \infty$
  $\mathrm{Ext}^i(E,F)^* = \mathrm{Ext}^{3-i}(F,E)$.

  $+\ cl : K_0(C) \to \Lambda$

  $\underline{\underline{\text{st.}}}$ $\langle E, F \rangle := \sum (-1)^i \ \mathrm{rk}\ \mathrm{Ext}^i(E,F)$ factors through a skew-symm. pairing $\langle \cdot, \cdot \rangle$ on $\Lambda$.

  $D$ comm. ring assoc. to $k$, containing an invertible element "$q^{1/2}$"

  quantum torus of $\Lambda$ $:= \bigoplus_{\gamma \in \Lambda} D . e_\gamma$, $\quad e_{\gamma_1} \cdot e_{\gamma_2} = q^{\langle \gamma_1, \gamma_2 \rangle / 2} e_{\gamma_1 + \gamma_2}$

<u>Plan:</u> ‖ Motivic Hall alg$(C)$ $\longrightarrow$ quantum torus$(\Lambda)$ $\xrightarrow[\text{limit } q \to 1]{\text{semi-classical}}$ DT invariants of $C$.

- What is $D$? <u>"matrices of nearby cycles"</u>

  For $k = \mathbb{C}$: $\exists$ map $D \longrightarrow \left( \text{Algebra } \sum_{\substack{p, q \in \mathbb{Q} \\ p - q \in \mathbb{Z}}} c_{pq}\, z_1^p z_2^q \ (c_{pq} \in \mathbb{Q}) \right) [(1 - z_1^n z_2^n)^{-1}]$ $\forall n \geq 1$

  st. $q^{1/2} \mapsto z_1^{1/2} z_2^{1/2}$

st. for $X$ smooth proj. var., we get an elt which maps to

$$\longrightarrow \sum_{p,q} \mathrm{rk}\, H^{p,q} (-1)^{p+q} z_1^p z_2^q$$

for $G$ acting on $X$, $X/GL(n) \longrightarrow \dfrac{H(X)}{\prod_{i=1}^{r-1} \left(z_1^n z_2^n - z_1^i z_2^i\right)}$

If $X \circlearrowleft M$ autom. of finite order : $\forall p, q$, $H^{p,q} = \bigoplus_{\substack{\lambda \in \mathbb{Q}/\mathbb{Z} \\ \text{eigenval. of } M}} H^{p,q,\lambda}$

$$\sum_{p,q} \mathrm{rk}\, H^{p,q,0} (-1)^{p+q} z_1^p z_2^q$$

$$+ \sum_{\substack{0<\lambda<1 \\ p,q}} \mathrm{rk}\, H^{p,q,\lambda} (-1)^{p+q} z_1^{p+\lambda} z_2^{q+1-\lambda}$$