M. Kontsevich – 22/1/08 – Motivic Donaldson Thomas invariants I (w/ Soibelman)

X smooth compact CY 3-fold

Counting problems:

1. GW invariants $\quad N_{g,d}^{GW} \in \mathbb{Q}$ $\quad g \geq 0$ genus, $d \in H_2(X, \mathbb{Z})$

$$\dashrightarrow \quad F = \exp\Big(\sum_{g,d} \lambda^{2g-2} q^d N_{g,d}^{GW} \Big) \quad \text{generating } f^n \quad (\text{cv unclear})$$

$|\lambda| \ll 1$, $q \in \mathrm{Hom}(H_2, \mathbb{C}^*)$

is a holom. function on a certain formal alg. scheme.

2. DT invariants

Basic idea: $\mathcal{E}$ holom. v.b. $/X$

Deforms of $\mathcal{E}$:= $R\mathrm{Hom}(\mathcal{E}, \mathcal{E})$ as dg Lie algebra

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ \mathrm{Ext}^0 & \mathrm{Ext}^1 & \mathrm{Ext}^2 & \mathrm{Ext}^3 \end{array}$$

Ext^0: usually $\simeq \mathbb{C}\, id_{\mathcal{E}}$

$\underbrace{\mathrm{Ext}^1 \quad \mathrm{Ext}^2}$: ideal situation for deform^n theory

$\mathrm{Ext}^3 \simeq (\mathrm{Ext}^0)^*$ in CY case

$\rightarrow$ shorten deform^n theory by truncating to $\tau_{\leq 2} / \mathbb{C} \cdot id_{\mathcal{E}}$.

Then lives in degrees 0, 1, 2.

Only one case when moduli stack is <u>proper</u> & $\mathrm{End}(\mathcal{E}) \simeq \mathbb{C}\, id_{\mathcal{E}}$:

$\mathcal{E}$ coherent sheaf = <u>ideal sheaf</u> of <u>subscheme</u> $Y \subset X$, $\dim Y \leq 1$

$$\mathcal{E}_{|X \setminus Y} \cong \mathcal{O}_{X \mid X - Y}.$$

☆ <u>Hope</u>: extend DT invts N_γ^{DT} $\quad \gamma \in K_0^{top}(X) \underset{\text{up to torsion}}{\approx} H^{even}(X, \mathbb{Z})$

• <u>Conj</u> (Maulik-Nekrasov-Okounkov-Pandharipande)

There are better invts = Gopakumar-Vafa invts $N_{g,d}^{GV} \in \mathbb{Z}$ $\quad d \neq 0$

(tentative def^n: pick a <u>generic</u> a.c.s. J on X tamed by ω & count J-holom. curves $\subset X$ with coeffts ± 1)

... and expect both N^{GW} and N^{DT} can be related by explicit formulas to N^{GV}.

Special Lagrangian submanifolds:

$L \subset X$ Lag., $\quad \operatorname{Arg} \Omega^{3,0}_{|TL} = \varphi \in \mathbb{R}/2\pi\mathbb{Z}$ constant

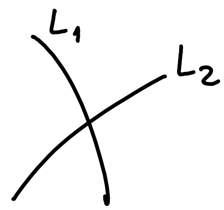
deformation theory is unobstructed & governed by $H^*(L,\mathbb{R})$
ie. degs. 0 and 1 only.

Would like invts counting such L?
e.g. # of SLag spheres in a given class of $H_3(X)$.

This should be <u>mirror</u> to N^{DT}_γ $\mathbb{C}$-curve counting problems?

⚠ These are not top. invariants — depend on <u>complex structure</u>!

Indeed:



$L_1, L_2 \approx S^3$ SLags with phases $\varphi_1 \neq \varphi_2$:
$$\int_{L_i} \Omega^{3,0} = e^{i\varphi_i} \operatorname{vol}(L_i)$$

As we vary the $\mathbb{C}$ structure, φ_1, φ_2 change and we have a codim. 1 <u>wall</u> where $\varphi_1 = \varphi_2$; typically:

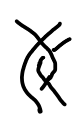
L_1, L_2	L_1, L_2, <u>and</u> $L_1 \# L_2$.

↑
wall: L_1, L_2, and formally $L_1 \cup L_2$ special Lag.

$\Rightarrow$ <u>Naive wall-crossing formula</u>:

$N^{DT}_{\gamma_1+\gamma_2}$ changes by $N^{DT}_{\gamma_1} N^{DT}_{\gamma_2} \langle \gamma_1, \gamma_2 \rangle$

↗ intersection pairing on H_3.
(#choices of attaching points to do $L_1 \# L_2$?)

However: if $|L_1 \cap L_2| = 2$, could also take  $L_1 \# L_2 \# L_1$, etc ...

$\Rightarrow$ there are also higher order terms !!!

Conjecture: for 3-dim^{l} CY X, $\exists$ numbers $N^{DT}_\gamma \in \mathbb{Q}$, $\gamma \in H_3(X,\mathbb{Z})$, which 1) do not depend on Kähler class
2) change under variation of $\mathbb{C}$ str. according to our wall-crossing formula.

$\Lambda = H_3(X,\mathbb{Z})/\text{Torsion}$, $\langle \cdot,\cdot \rangle : \Lambda \times \Lambda \to \mathbb{Z}$ intersection pairing

Pick $\|\cdot\|$ any norm on $\Lambda \otimes \mathbb{R}$; and let $Z = \int \Omega^{3,0} : \Lambda \to \mathbb{C} = \mathbb{R}^2$.

- key Property: $\exists c > 0$ s.t. if $N^{DT}_\gamma \neq 0$ then $\|\gamma\| \leq c\,|Z(\gamma)|$.

  Indeed: if $N^{DT}_\gamma \neq 0$ then $\exists L$ SLag s.t. $[L] = \gamma$.

  Then $|Z(\gamma)| = \left|\int_L \Omega^{3,0}\right| = \mathrm{vol}(L)$

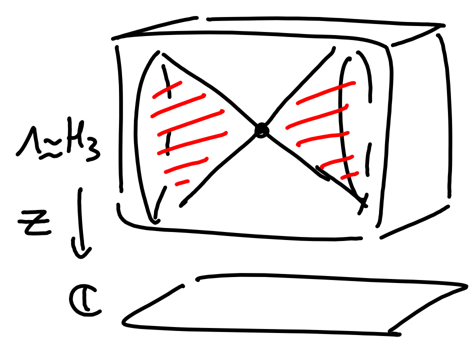
  But given a closed 3-form on X, $\int_L \text{form} \leq \underset{\uparrow}{\text{const}}\cdot\mathrm{vol}(L)$.

  L^∞-norm of form

  $\rightarrow$ so $\|\gamma\| \leq \underset{\uparrow}{\text{const}}\cdot\mathrm{vol}(L)$

  depending only on chosen norm $\|\cdot\|$.

Hence: support of DT inv't is contained in a cone



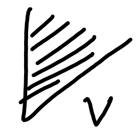
- For given $Z : \Lambda \to \mathbb{C}$ additive map,
  given a quadratic form Q on $\Lambda \otimes \mathbb{R}$ st. $Q|_{\ker Z \otimes \mathbb{R}} < 0$,
  say $N^{DT} : \Lambda \to \mathbb{Q}$ is controlled by Q if
  $$N^{DT}_\gamma \neq 0 \Rightarrow Q(\gamma) > 0$$
  (by above property, such a Q exists).

- Change Z continuously: $(Z_t)_{t\in[0,1]}$; assume $Q|_{\ker Z_t \otimes \mathbb{R}} < 0 \ \forall t$.

  $\rightsquigarrow$ now can ask for $N^{DT,t}_\gamma$ to be controlled by Q.

- Let $\mathfrak{g}_\Lambda = \bigoplus_{\gamma \in \Lambda} \mathbb{Q} \cdot e_\gamma$ with $[e_{\gamma_1}, e_{\gamma_2}] = (-1)^{\langle \gamma_1, \gamma_2 \rangle} \langle \gamma_1, \gamma_2 \rangle e_{\gamma_1 + \gamma_2}$

- Given Z, N_γ^{DT}, pick an angular sector $V \subset \mathbb{R}^2 = \mathbb{C}$, of angle $< 180°$



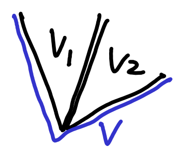
$\Rightarrow$ let $\mathfrak{g}_V = \prod_{\gamma \in \Lambda \cap \left(\text{Convex hull of } Z^{-1}(V) \cap Q^{-1}(\mathbb{R}_+)\right)} \mathbb{Q} e_\gamma$ Lie subalg. $\subset \mathfrak{g}_\Lambda$

and $G_V = \exp(\mathfrak{g}_V)$ Lie group

Then to V we associate an element $A_V \in G_V$:

$$A_V = \prod_{\substack{\text{rays } \ell \subset V \\ \text{in clockwise order}}} \exp\left(\sum_{\substack{\gamma \\ Z(\gamma) \in \ell}} N_\gamma^{DT} \left(\sum_{k \geq 1} \frac{e_{k\gamma}}{k^2} \right) \right)$$

In particular if $V = \underset{\text{left}}{V_1} \sqcup_\partial \underset{\text{right}}{V_2}$



then $A_V = A_{V_1} \cdot A_{V_2}$ $(G_{V_1}, G_{V_2} \subset G_V)$.

<u>Axiom</u>: <u>A_V stays constant as long as $\forall \gamma : Q(\gamma) > 0$, $Z(\gamma) \notin \partial V$.</u>

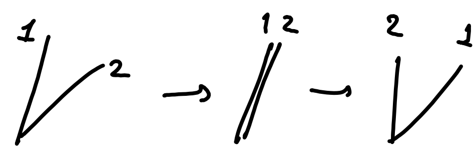
- Consider only a 2D lattice $\Lambda_2 \simeq \mathbb{Z}^2 \subset \Lambda$:

wall is where $\Lambda_2 \xrightarrow[Z]{} \mathbb{R} e^{i\varphi}$ instead of rank 2.

If $\langle \cdot, \cdot \rangle_{|\Lambda_2} = 0$ then numbers won't change.

Otherwise, change as the 2 generators γ_1, γ_2 of Λ_2 get "reversed"

$\rightarrow$ <u>formulation of wall-crossing restricted to Λ_2:</u>



Consider $\mathbb{Q}[[x,y]]$ with $\{x, y\} = k\, xy$, $k \in \mathbb{Z}$, $\neq 0$

where <u>$k = \langle \gamma_1, \gamma_2 \rangle$</u>

$\forall a,b \geq 0$, let $T_{a,b}^{(k)} \in \mathrm{Aut}\, \mathbb{Q}[[x,y]]$, defined by

$$T_{a,b}^{(k)} : \quad x \longmapsto (1 \pm x^a y^b)^{kb}\, x \qquad (\text{sign: } -(-1)^{kab})$$
$$\qquad\qquad y \longmapsto (1 \pm x^a y^b)^{ka}\, y$$

Then $\boxed{\prod_{a/b \uparrow} T_{a,b}^{N^{DT,\,left}_{a\gamma_1 + b\gamma_2}} = \prod_{a/b \downarrow} T_{a,b}^{N^{DT,\,right}_{a\gamma_1 + b\gamma_2}}}$

Eg: $k=1$: $\quad T_{1,0}\, T_{0,1} = T_{0,1}\, T_{1,1}\, T_{1,0}$

(as above: $L_1, L_2 \;\underset{\text{wall}}{\Big|}\; L_1, L_2, L_1 \# L_2$)

$k=2$: $\quad T_{1,0}\, T_{0,1} = T_{0,1}\, T_{1,2}\, T_{2,3} \cdots \underbrace{T_{1,1}^{-2}\, T_{2,2}^{+1}}_{\text{middle terms}} \cdots T_{3,2}\, T_{2,1}\, T_{1,0}$

etc...

these formulas encode wall-crossing for a given pair L_1, L_2 for various value of $\langle \gamma_1, \gamma_2 \rangle$. $\quad (N_{\gamma_1} = 1,\ N_{\gamma_2} = 1)$.

(in particular, if know N^{DT} on one side of wall then get N^{DT} on the other side!)