M. Gualtieri - 22/1/08 - Generalized complex structures on 4-mflds (w/ Cavalcanti)

A 4D generalized $\mathbb{C}$ structure is given by a line subbundle

$$K \subset \Lambda^{ev} T^* \otimes \mathbb{C} = \Lambda^0_{\mathbb{C}} \oplus \Lambda^2_{\mathbb{C}} \oplus \Lambda^4_{\mathbb{C}}$$

st. any generator $\rho = \rho_0 + \rho_2 + \rho_4$ of K satisfies

$$\begin{cases} ① \ \langle \rho, \rho \rangle = 0 \quad \text{where } \langle \rho, \sigma \rangle := \rho_0 \sigma_4 - \rho_2 \wedge \sigma_2 + \rho_4 \sigma_0 \in \Lambda^4 \\ ② \ \langle \rho, \bar{\rho} \rangle \neq 0 \quad (\text{nondegeneracy}) \\ ③ \ d\rho = (X + \xi) \cdot \rho \quad \text{for some } X + \xi \in C^\infty(T \oplus T^*) \\ \qquad \qquad (\text{integrability}) \end{cases}$$

This comes from thinking of $\Lambda^\bullet T^*$ as <u>spinors</u> for $T \oplus T^*$.

<u>Examples</u>: ① if $K \subset \Lambda^2 T^* \otimes \mathbb{C}$:

$\rho \wedge \rho = 0 \iff \rho$ is decomposable

$\rho \wedge \bar{\rho} \neq 0 \Rightarrow \rho$ determines an almost-cx. structure for which $K = \Omega^{2,0}$

(3) $\Rightarrow$ the a.c.s. is integrable.

② $K = \text{span}(\rho = e^{i\omega} = 1 + i\omega - \frac{1}{2}\omega \wedge \omega)$, $\omega \in \Omega^2(M)$ <u>symplectic</u>

key property distinguishing ② from ①: K has nontrivial projection to Ω^0.

③ $\mathbb{C}^2$, coords. (z_1, z_2), $\rho = z_1 + dz_1 \wedge dz_2$.

- $\langle \rho, \rho \rangle = dz_1 \wedge dz_2 \wedge dz_1 \wedge dz_2 = 0$ ✓
- $\langle \rho, \bar{\rho} \rangle = dz_1 \wedge dz_2 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \neq 0$ ✓
- $d\rho = dz_1 = \left(-\frac{\partial}{\partial z_2}\right) \cdot \rho$ ✓

$\rightarrow$ When $z_1 \neq 0$, $\rho = z_1\left(1 + \frac{dz_1 \wedge dz_2}{z_1}\right) = z_1 e^{dz_1 \wedge dz_2 / z_1}$

$\Rightarrow K = \langle e^{B + i\omega} \rangle$ where $B + i\omega = \frac{dz_1 \wedge dz_2}{z_1}$

B-field transform of sympl. structure.

The sympl. form can be written as

$$\omega = d\log r \wedge du + d\theta \wedge dv \qquad \text{where } z_1 = re^{i\theta}, \quad z_2 = u + iv.$$

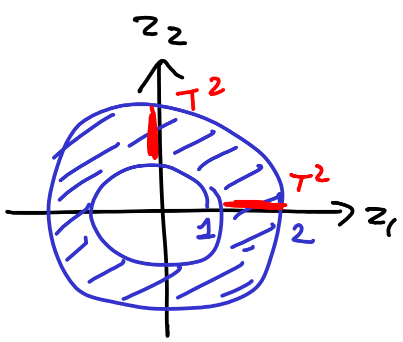
$\to$ When $z_1 = 0$, $\rho = dz_1 \wedge dz_2$ defines a complex structure.

This kind of type-change behavior is generic.

(always happens in this way: generically we have $B + i\omega$;
$\mathbb{C}$ locus = where Λ^0-component of ρ vanishes
= along a submanifold)

<u>Note</u>: this example is inv't under z_2-translations, so descends to a GCS on $T^2 \times \mathbb{R}^2$.

④ $S^1 \times S^3 = \mathbb{C}^2 \setminus \{0\} / (z \mapsto 2z)$



$\rho = z_1 z_2 + dz_1 \wedge dz_2$

$K = \langle \rho \rangle$ is inv't under $z \mapsto 2z \Rightarrow$ well-def'd on $S^1 \times S^3$
& defines a GCS by same argument

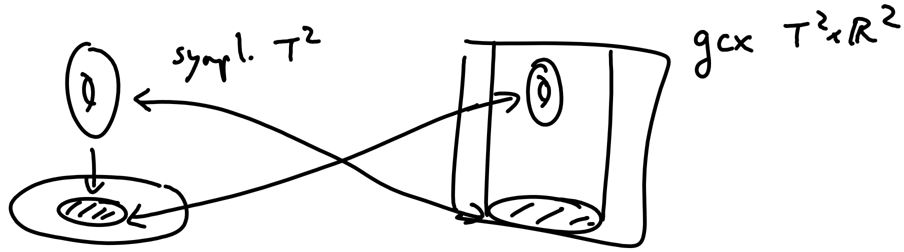
Type-change locus = $\begin{cases} z_1 = 0 & \cong T^2 \\ \text{and} & \\ z_2 = 0 & \cong T^2 \end{cases} \Rightarrow$ pair of T^2's.

<u>Q</u>: what types of 4-mflds have GCS's ??
Come up with examples? <u>surgeries</u> for GC 4-mflds?

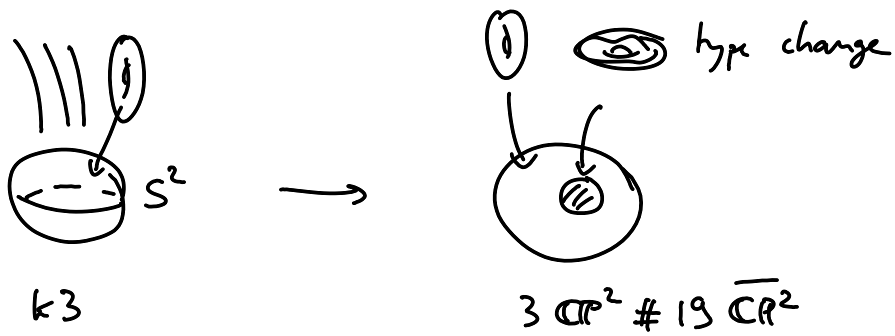
$\boxed{1a}$ <u>Logarithmic surgery</u>: a nbd of a square zero symplectic 2-torus in a sympl. 4-mfld is $\cong T^2 \times D^2 \subset T^2 \times \mathbb{R}^2$

Replace a nbd. with the gcx $T^2 \times D^2$:

⚠ topologically this is a <u>0-surgery</u> on the T^2.



<u>Ex:</u> take an elliptically fibered K3, and replace a fiber nbd with the gcx $T^2 \times D^2$:



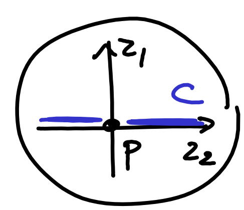
Gompf-Mrowka $\Rightarrow$ $SW(3\,\mathbb{CP}^2 \# 19\,\overline{\mathbb{CP}}^2) = 0$

$\Rightarrow$ cannot support a sympl. structure

$(b_1 = 0)$ $\Rightarrow$ cannot be complex either.

$\boxed{2}$ <u>Blow up / blow down</u>:

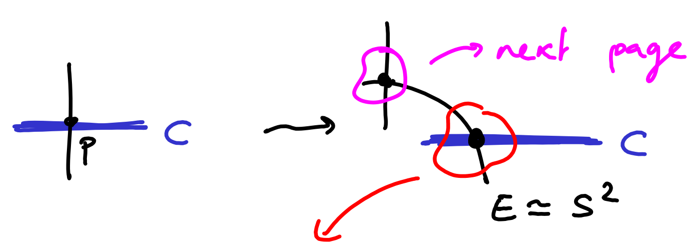
Let $p \in C$, $C =$ type change locus



near p, $K = \langle \rho \rangle$, $\rho = z_1 + dz_1 \wedge dz_2$.

(using Darboux-Moser style argument).

Blowing up p:



In new coordinates: near proper transform of C,

$\tilde{\rho} = \tilde{z}_1 z_2 + d(\tilde{z}_1 z_2) \wedge z_2$ $\quad$ [$\tilde{z}_1 =$ coord. along E ; $z_1 = \tilde{z}_1 z_2$
$z_2 =$ coord. along C]

$= \tilde{z}_1 z_2 + z_2\, d\tilde{z}_1 \wedge dz_2$

↪ proportional to $\tilde{z}_1 + d\tilde{z}_1 \wedge dz_2$

In the other chart near proper transform of z_1-axis,

$$\tilde{\rho} = z_1 + dz_1 \wedge d(z_1 \tilde{z}_2) \qquad (\tilde{z}_2 = \text{coord. along } E,\ z_2 = z_1 \tilde{z}_2)$$
$$= z_1 + z_1\, dz_1 \wedge d\tilde{z}_2$$
$$\cong 1 + dz_1 \wedge d\tilde{z}_2 = e^{dz_1 \wedge d\tilde{z}_2} \qquad \text{(no type change).}$$

<u>So</u>: type change locus is the <u>proper transform</u> of C.

<u>Moreover</u>: E is <u>Lagrangian</u> (except where it hits C)

since $dz_1 \wedge d\tilde{z}_2 = B + i\omega$ vanishes on $E = \{z_1 = 0\}$.

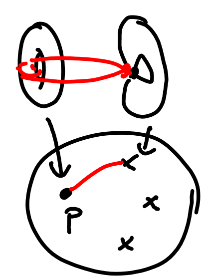
$\Rightarrow$ E is a 2-sphere of self-int. (-1), and Lagrangian except where it intersects C.

(this is an example of a GCx "brane")

<u>Thm</u> (Cavalcanti-G.):

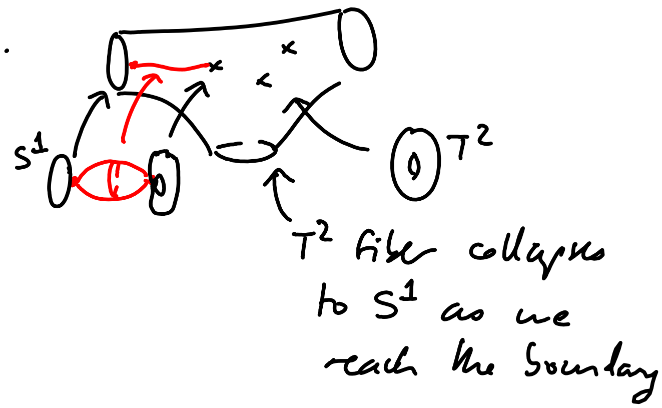
This is a well-def^{d} operation, and if we find such a (-1)-sphere we can blow it down.

<u>Ex</u>: in a symplectic fibration, e.g. K3, we have Lagrangian vanishing cycles:



Under surgery in a nbd of fiber above p, one direction of the T^2-fiber gets collapsed; if the vanishing cycle matches with this, then it becomes a (-1)-sphere which intersects the type change locus transversely once.

In fact, we end up with a "generalized sympl. fibration" over a surface w/ boundary.



On K3, $\exists$ ell. fibration with I_{19} fiber

$\Rightarrow$ can get 19 vanishing cycles w/ same loop a collapsed

Doing one log surgery, get $3\mathbb{CP}^2 \# 19 \overline{\mathbb{CP}^2} \supset 19$ (-1)-S^2 branes

$\Rightarrow$ blowing them down, get a gcx str. on $3\mathbb{CP}^2$.