M. Aldi – 25/11/08 – A-branes & (noncomm.) coord. rings

M sympl. mfld, $c_1 = 0$ $\xleftrightarrow{\text{M.S.}}$ W mirror mfld, cplex (projective)

with $D^b \operatorname{Fuk}(M) \cong D^b \operatorname{Coh}(W)$

$\underline{Q^n}$: how to find the mirror CY ?

- $\mathcal{O}_W$ should correspond to some Lagrangian submfld $L_0 \subset M$
- $- \otimes \mathcal{O}_W(1)$ functor on $D^b W$ should come from a sympl. $\rho : M \circlearrowleft$ so that on $D^b \operatorname{Fuk}(M)$ we get $L \mapsto \rho(L)$.

Consider a proj. mfld $W = \operatorname{Proj}(R)$,

R graded ring $\quad R = \bigoplus_{k \geq 0} \operatorname{Hom}(\mathcal{O}_W, \mathcal{O}_W(1)^{\otimes k})$

$\Rightarrow$ given M, $L \subset M$ Lagr., $\rho : M \to M$ symplecto.,

should look at $\quad R = \bigoplus_{k \geq 0} \operatorname{Hom}(L_0, \rho^k(L_0))$

and set $W = \operatorname{Proj}(R)$.

$\underline{\text{Ex}}$: The elliptic curve – Polishchuk-Zaslow

$M = T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with $\int_{T^2} B + i\omega = \tau$:

take $\begin{cases} L = x\text{-axis} \\ \rho : (x,y) \mapsto (x, y+3x) \end{cases}$

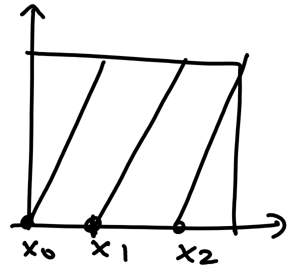
Let $R = \bigoplus_{k=0}^{\infty} D\operatorname{Fuk}^0_M(L, \rho^k L) \underset{\text{as vect. space}}{=} \bigoplus_{k=0}^{\infty} \mathbb{C}\langle L \cap \rho^k L \rangle$

$\underline{\text{Thm}}$: (Zaslow):

$$R = \mathbb{C}[x_0, x_1, x_2] \big/ (x_0^3 + x_1^3 + x_2^3 + \lambda x_0 x_1 x_2 = 0)$$

where λ is st. $\operatorname{Proj}(R) \cong \mathbb{C}/\mathbb{Z} \oplus \tau\mathbb{Z}$

Idea pf:



$R := $ as above

$\Rightarrow R_1 = \mathbb{C}\langle L \cap \rho L\rangle = \mathbb{C}\langle x_0, x_1, x_2\rangle$

and $\dim R_k = \#(L \cap \rho^k L) = 3k$

Product structure? $\quad x_0 x_1 = x_0 \bullet_{\mathrm{Fuk}} \rho(x_1)$

$$= \sum_{n\in\mathbb{Z}} e^{2\pi i\left(\frac{1}{3}+n\right)^2 + \cdots} \cdots$$

theta-function

look at all triangles

Check $R_2 \simeq \mathrm{Sym}^2 R_1$

But now $\mathrm{Sym}^3 R_1$ has rank 10, R_3 has rank 9

$\Rightarrow \exists$ relation! Compute it & get the result.

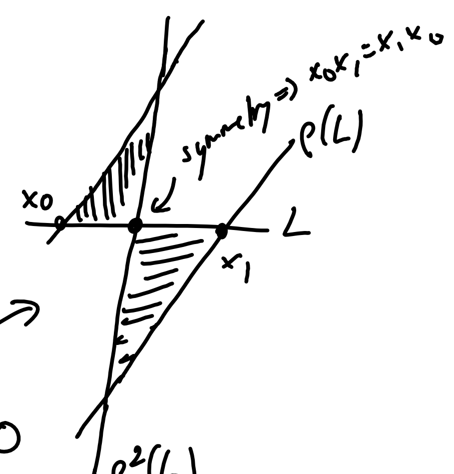
$$x_0^3 + x_1^3 + x_2^3 = \lambda x_0 x_1 x_2 .$$

$M,\ L\subset M,\ \rho \circlearrowright M \rightsquigarrow R.$

- Look again at $M = \mathbb{R}^2/\mathbb{Z}^2$, $\rho(x,y) = (x, y+mx)$ $\quad m\in\mathbb{Z}_{>0}$
  - for $m\geq 3$, $\quad W = \mathrm{Proj}(R) \hookrightarrow \mathbb{P}^{m-1}$ (elliptic curve)
  - for $m=2$ $\quad \hookrightarrow \mathbb{P}(1,1,2)$
  - for $m=1$ $\quad \hookrightarrow \mathbb{P}(1,2,3)$.

- But if we look at $\rho(x,y) = (x+b,\ y+3x)$:

$$L\cap\rho(L) = \{x_0(b), x_1(b), x_2(b)\}$$



For $b=0$, get $x_i x_j = x_j x_i$ (symmetry)

Commutativity follows from equality of areas

For $b\neq 0$, $p(b)\, x_0(b)^2 + q(b)\, x_1(b) x_2(b) + r(b)\, x_2(b) x_1(b) = 0$

+ cyclic permutations

$\Rightarrow$ now get a Sklyanin algebra

$$\mathbb{C}\{x,y,z\} \Big/ \begin{matrix} p(b)x^2 + q(b)yz + r(b)zy = 0 \\ \text{\& cyclic permutations} \end{matrix} \cong Q_3(E_\tau, b)$$

so now $W \hookrightarrow \mathbb{P}^2_{NC}$ (and in fact W is precisely the set of commutative points)

<u>Surfaces</u>:

$M = \mathbb{R}^4/\mathbb{Z}^4$, coords. x_1, y_1, x_2, y_2

$B + i\omega = \tau_1\, dx_1 \wedge dy_1 + \tau_2\, dx_2 \wedge dy_2 + \tau_3 (dx_1 \wedge dy_2 + dx_2 \wedge dy_1)$

$L = (\mathbb{R}^2/\mathbb{Z}^2)_{x_1 x_2} = \{y_1 = y_2 = 0\}$ $\quad\quad \operatorname{Im}\begin{pmatrix} \tau_1 & \tau_3 \\ \tau_3 & \tau_2 \end{pmatrix} > 0.$

$\rho(x_i, y_i) = (x_i, y_i + m x_i)$

- $m = 3$: $W \hookrightarrow \mathbb{P}^8$
- $m = 2$: $\tilde{M} = M/i$, $i(x_i, y_i) = (-x_i, -y_i)$ sympl. Kummer

.... $\to$ get Kummer surface. $\to$ ....

<u>Thm</u>: Every singular Kummer surface in $\mathbb{P}^3$ can be obtained as $\operatorname{Proj}(\tilde{R})$.

If we add a translation term: $(x_i, y_i) \mapsto (x_i + b_i, y_i + 3x_i)$

$\Rightarrow$ get $W \hookrightarrow$ noncommutative $\mathbb{P}^8$

(<u>new</u> examples of NC $\mathbb{P}^8$?)