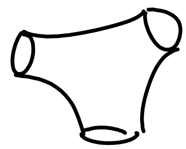
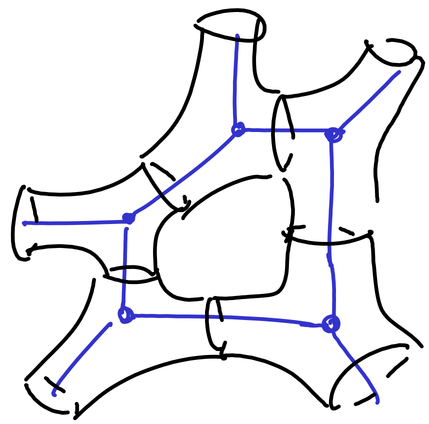
M. Abouzaid - 25/1/08 - HMS & pairs of pants

- Motivation: mirror symmetry for a hypersurface $\subset (\mathbb{C}^*)^n$, as seen through tropical geometry. Tropical geom. gives a decompⁿ into "pairs of pants", and would like to build $\left\{ \begin{array}{l} \text{Fukaya category} \\ \text{mirror manifold} \end{array} \right\}$ by using restrictions to the pairs of pants, gluing, ...
  ↪ requires exact Lagrangians ⇒ hence open sympl. mflds

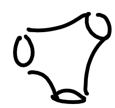
(I) (Joint w/ Paul Seidel)

- Pairs of pants $:= \mathbb{CP}^n \setminus (n+2)$ hyperplanes in general position
  $= (\mathbb{C}^*)^n \setminus$ hyperplane in general position

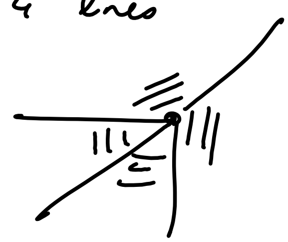
Ex: • $\mathbb{CP}^1 \setminus 3$ pts $=$ 



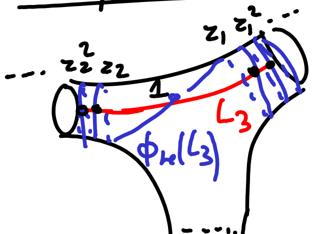
Tropical description of a curve:

vertex $\longleftrightarrow$ 

edge $\longleftrightarrow$ 

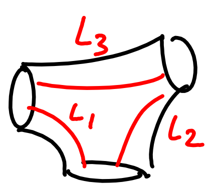
• in dim. 2, pair of pants $\mathbb{CP}^2 \setminus 4$ lines is tropically $\simeq$  $\subset \mathbb{R}^3$

- Wrapped Floer homology:



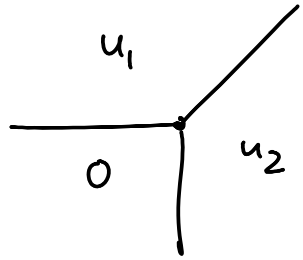
ϕ_H Hamiltonian flow of a function which grows fast enough at ∞

$$HW^*(L_3, L_3) \underset{\text{def}}{:=} HF^*(L_3, \phi_H(L_3)) \underset{\text{calculation}}{=} \mathbb{C}[z_1, z_2]/(z_1 z_2)$$

Similarly can look at L_1 & L_2 

$$= \mathbb{C}[z_1, z_2, z_3]/(z_1 z_2, z_3)$$

- <u>Mirror?</u>



$\subset \mathbb{R}^2_{(u_1,u_2)}$

tropical pair of pants := defined by $\varphi = \max(0, u_1, u_2)$ ($\longleftrightarrow$ "$1 + x_1 + x_2 = 0$")
(in general: convex piecewise linear function φ).

$\updownarrow$ tropical mirror

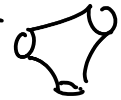
$\{ t \geq \varphi(u_1, u_2) \} \subset \mathbb{R}^3_{(u_1,u_2,t)}$

describes an <u>open</u> <u>toric variety</u>, with a map to $\mathbb{A}^1$ given by t-direction.

Here: get $\mathbb{C}^3$, $W = z_1 z_2 z_3$.

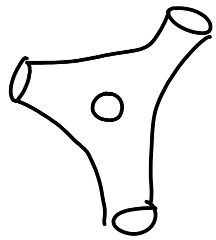
<u>Claim:</u> LG model $\mathbb{C}^3 \xrightarrow[W = z_1 z_2 z_3]{} \mathbb{A}^1$ is mirror to pair of pants.

- Check: L_3 as above $\longleftrightarrow$ M.F. $M_3 = \{ \mathbb{C} \xrightarrow{z_1 z_2} \mathbb{C} \xrightarrow{z_3} \mathbb{C} \}$
  similarly L_1, L_2

<u>Thm:</u> Cat. in Fuk() gen^{d} by L_1, L_2, L_3
$\cong$ Cat. in MF$(W = z_1 z_2 z_3)$ gen^{d} by M_1, M_2, M_3

$\star$ Hypersurface in $(\mathbb{C}^*)^2$ $\longleftrightarrow$ Laurent polynomial
$\downarrow$
Newton polytope (convex hull of exponents)
$\downarrow$
Toric variety with a map to $\mathbb{A}^1$ $\longleftrightarrow$ Overgraph of φ in $\mathbb{R}^3$

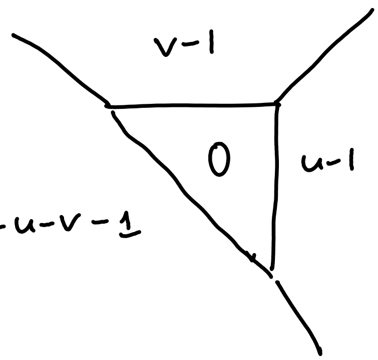
Ex:

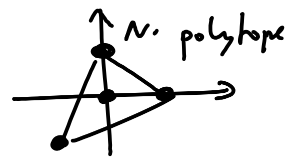


$1 + x + y + \frac{1}{xy}$

$\downarrow$

$\varphi =$





$\downarrow$

3D toric var. + w glued out of 3 pieces $\cong (\mathbb{C}^3, xyz)$
(one for each vertex).

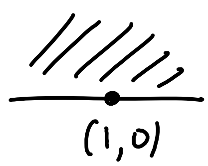
(II) (w/ Katzarkov & Auroux) SYZ construction:
(motivation: mirror of genus 2 curve?)

Thm (Orlov): $C \subset X \Rightarrow D^b\mathrm{Coh}(C) \subset D^b\mathrm{Coh}(Bl_C(X))$.

So: C = genus 2 curve $\subset X$, st. $\exists\ T^n \subset X \to B$ SYZ Slag fibration

$\rightsquigarrow$ try to construct Slag fibration on $Bl_C(X)$
then apply SYZ to get the mirror.

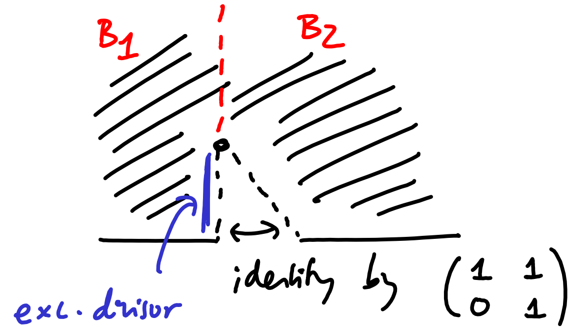
Example 1: blowup $p = (1,0) \in \mathbb{C}^* \times \mathbb{C}$
$\downarrow$ moment map gives toric Slag fibration
$\mathbb{R} \times \mathbb{R}_+$

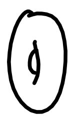


symplectic area of blowup

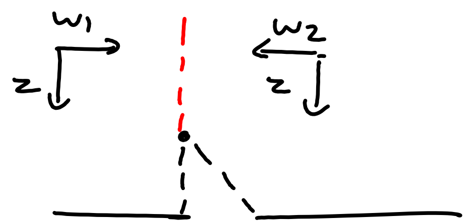
- The blowup $Bl_p^{\varepsilon}(\mathbb{C}^*\times\mathbb{C})$ carries a sing. toric fibration:

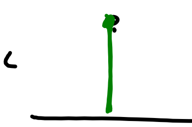
base $B = B_1 \cup B_2$



Fiber is  except at • it's 

- SYZ Mirror:



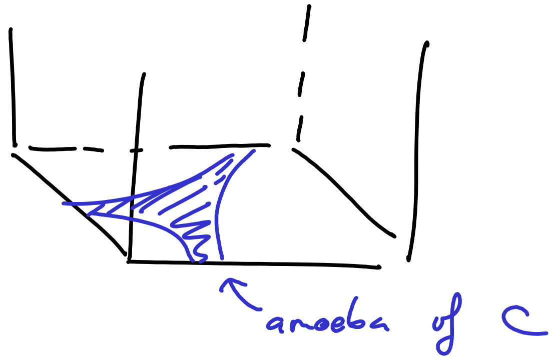
Mirror of 1st half has coords. (w_1, z)
and superpotential z (since height = area of disc )
W counts discs with exp(-Area).

Mirror of 2nd half has coords. (w_2, z).
and superpotential z.

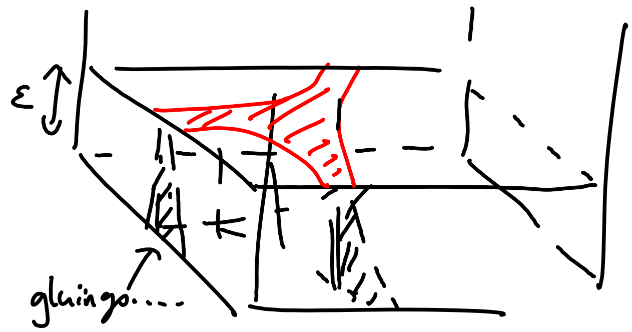
Topological gluing in the cut is $w_1 = w_2^{-1} z$.
This is not compatible with Floer theory.

Correct the gluing (cf. Kontsevich-Soibelman, Gross-Siebert)
$\Rightarrow$ get: $\{(z, w_1, w_2) \in \mathbb{C}^3 \,/\, w_1 w_2 = 1 + z\}$
with superpotential z
$\Longleftrightarrow$ $\mathbb{C}^2_{w_1,w_2}$ with map to $\mathbb{A}^1$ given by $w_1 w_2 - 1$
– similar to previous constr. for mirror of pt!

Example 2: blow up $C = \{1+x+y=0\} \subset \underbrace{(\mathbb{C}^*)^2 \times \mathbb{C}}_{\text{Lagr. torus fibration over } \mathbb{R}^2 \times \mathbb{R}_+}$

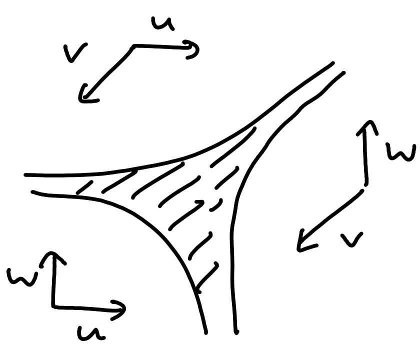


Then $Bl^{\varepsilon}_{C}((\mathbb{C}^*)^2 \times \mathbb{C})$ has a Lag. torus fibration with sings.:



⚠ discr. locus is codim. 1, <u>thickening</u> of codim. 2 tropical case. (cf. <u>Joyce</u>).

Then: seen from above:



+ global z coordinate

<u>gluing</u> between these parts $\Rightarrow$ $uvw = 1+z$

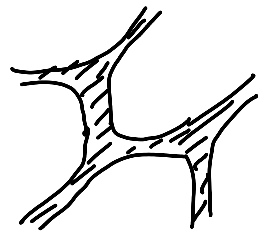
[comes from homotopy method in Lagr. Floer homology]

$\Rightarrow$ mirror is $\{(u,v,w,z) \in \mathbb{C}^4 \,/\, uvw = 1+z\}$

with superpotential $= z$

$\longleftrightarrow$ same as before!

Now if we have a more complicated curve,
look at its amoeba (when close enough to tropical)



Then glue pieces as above

$\Rightarrow$ same gluing result as by previous method !!!