J. Solomon - 22/1/08 - Open GW invariants

Goal: mediate between GW invts & Fukaya category : **OGW** invts

→ definition : for → target dim. 4 or 6
→ target dim. 0

In dim. 4/6: symmetry of (X, L) $L \subset (X, \omega)$ Lagrangian
= antisympl. involution
OGW = intersection theory on moduli space w/ boundary

→ structure : for → CY case : "KMS"
→ Fano case : "WDVV / integrable systems"

→ calculate: open-closed relations

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dim $X = 4$:

- **closed**: $GW_{g,d,n} : H^*(X, \mathbb{Q})^{\otimes n} \to \mathbb{Q}$
  $d \in H_2(X, \mathbb{Z})$
  $\alpha_i \in H^*(X)$, $A_i \subset X$ PD to α_i
  $\to \langle \alpha_1 \dots \alpha_n \rangle^c_{g,d} = GW_{g,d}(\alpha_1 \dots \alpha_n)$
  $=$ # stable curves of deg. d genus g passing through A_i

- $\phi : X \to X$, $\phi^* \omega = -\omega$, $\phi^2 = id$, $L = Fix(\phi)$
  $H^*_\phi(X) = (-1)^{*/2}$-eigenspace of ϕ^* on $H^{even}(X)$
  (ie. antiinvt part of $H^2, \dots$)
  $H^\phi_*(X)$ = same for homology

$OGW_{d,k,l} : H^*_\phi(X, \mathbb{Q})^{\otimes l} \to \mathbb{Q}$

genus 0
$d \in H_2^\phi(X, L)$

$\alpha_i \in H^*_\phi(X, \mathbb{Q})$, $\phi(A_i) = A_i$; $y_1 \dots y_k \in L$ points

$$\Rightarrow \langle \alpha_1 \dots \alpha_l \underbrace{\sigma^k}_{\text{point class on } \partial} \rangle_d := OGW_{d,k,l}(\alpha_1 \dots \alpha_l)$$

$= \# \left\{ \begin{array}{l} \text{open stable maps to } (X, L) \\ \text{degree } d \text{ passing through } A_i \\ \text{\& } \partial \text{ passes through } y_1 \dots y_k \end{array} \right\}$

Ex: $W_{d,m} = \frac{1}{2}\langle [2pt]^m \sigma^k \rangle_{\mathbb{CP}^2, d}$ $\quad 3d-1 = 2m+k$

$=$ # real rational curves through k real pts
m pairs of $\mathbb{C}$ conj. pts.

with signs: (Welschinger): sign $= (-1)^{\#\text{real isolated nodes}}$.

- Generating functions:

closed case: $\Phi(t) = \sum_{n,d} \frac{\langle \delta_t^{\otimes n} \rangle_d}{n!}$, where

$t = (t_i) \leftrightarrow$ basis $\delta_i \in H^*_\phi(X)$, and $\delta_t = \sum t_i \delta_i$

open case: $\sigma \rightsquigarrow$ new variable u

$$\Omega(t,u) = \sum_{\substack{k,l \\ \mu(d)\text{ even}}} \frac{u^k \langle \delta_t^{\otimes l} \sigma^k \rangle_d}{k!\,l!} + \sqrt{-1} \sum_{\substack{k,l \\ \mu(d)\text{ odd}}} \cdots\cdots$$

Maslov index

$$\langle\langle \delta_{i_1} \ldots \delta_{i_n} \rangle\rangle^c := \frac{\partial^n \Phi}{\partial t_{i_1} \ldots \partial t_{i_n}}$$

$$\langle\langle \delta_{i_1} \ldots \delta_{i_l} \sigma^k \rangle\rangle := \frac{\partial^{k+l} \Omega}{\partial t_{i_1} \ldots \partial t_{i_l} \partial u^k}$$

encodes
- all degrees d
- all insertion which include $\delta_{i_1}, \ldots, \sigma^k$, & anything else

Thm (WDVV, Tian): Let $g_{ij} = \langle \delta_i \cup \delta_j, [X] \rangle$, $(g^{ij}) = (g_{ij})^{-1}$

$$\langle\langle a, b, \delta_i \rangle\rangle^c g^{ij} \langle\langle \delta_j, c, d \rangle\rangle^c = \langle\langle b, c, \delta_i \rangle\rangle^c g^{ij} \langle\langle \delta_j, a, d \rangle\rangle^c$$

Corollary: recursive formula for $N_d =$ # deg. d rat'l curves $\subset \mathbb{P}^2$ through $3d-1$ pts.

Thm (S.):

(1) $$\sum_{ij} \langle\langle a, b, \delta_i \rangle\rangle^c g^{ij} \langle\langle \delta_j, c \rangle\rangle + \langle\langle a, b \rangle\rangle \langle\langle c, \sigma \rangle\rangle = \sum_{ij} \langle\langle c, b, \delta_i \rangle\rangle^c g^{ij} \langle\langle \delta_j, a \rangle\rangle + \langle\langle c, b \rangle\rangle \langle\langle a, \sigma \rangle\rangle$$

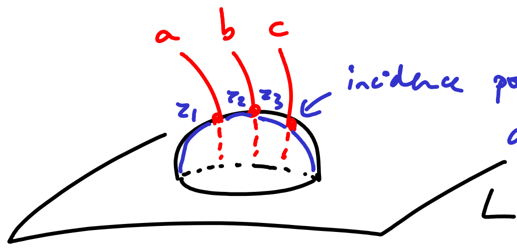
(2) $$\sum_{ij} \langle\langle a, b, \delta_i \rangle\rangle^c g^{ij} \langle\langle \delta_j, \sigma \rangle\rangle + \langle\langle a, b \rangle\rangle \langle\langle \sigma^2 \rangle\rangle = \langle\langle a, \sigma \rangle\rangle \langle\langle b, \sigma \rangle\rangle$$

Cor: (1) + (2) $\Rightarrow$ N_d & $W_{d,m}$ determined recursively from $W_{1,0}$

Also recover <u>Thm</u> (Itenberg-Kharlamov-Shustin)

$$\lim_{d\to\infty} \frac{\log W_{d,0}}{\log N_d} = 1 \qquad (\log N_d \sim 3d \log d)$$

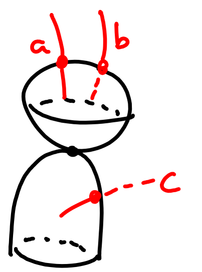
<u>Picture:</u>



incidence points are aligned along a hyperbolic geodesic and hyp. distance $\frac{d_H(z_1,z_2)}{d_H(z_2,z_3)} = k$ fixed const.

As $k\to 0$ this converges to either

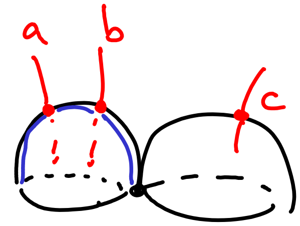
①



$((a,b,S_i))^c g^{ij} ((S_j,c))$

or

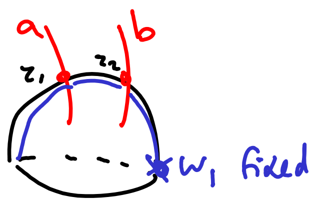
②



$((a,b))\,((c,\sigma))$

Formula (1) comes from comparing this with case $k\to\infty$
(similar with a/bc instead of ab/c).

Formula (2) is similar but with



requiring • alignment along a geodesic
• $d_H(z_1,z_2) = k$