J. Kamnitzer - 21/1/08 - Braid group actions on derived categories & Langlands

(w/ Sabin Cantis)

* Fix $m \geq 2$, let $p \in \mathbb{P}^1$

$$\Rightarrow \quad Gr_p^{(k)} := \left\{ (V, \varphi) \;/\; \begin{array}{l} V \text{ rank } m \text{ vect. bundle on } \mathbb{P}^1 \\ \varphi: V \to V_0 \text{ injective} \\ \quad\quad \text{trivial rank } m \text{ vect bundle} \\ \text{coker } \varphi \cong \mathcal{O}_p^{\oplus k} \end{array} \right\}$$

$0 < k < m$

(Beilinson-Drinfeld Grassmannian)

$$Gr_p^{(k)} \cong Gr(m-k, \mathbb{C}^m)$$

E.g., If $p = 0$, $z\mathbb{C}[z]^m \subset \Gamma(V, \mathbb{A}^1) \subset \mathbb{C}[z]^m = \Gamma(V_0, \mathbb{A}^1)$

codim. k subspace

- Varying p, get a bundle $Gr_{\mathbb{P}^1}^{(k)} \to \mathbb{P}^1$

* <u>More generally:</u> $Gr_{(p_1 \cdots p_n)}^{(k_1 \cdots k_n)} = \left\{ (V_1 \ldots V_n, \varphi_1 \ldots \varphi_n) : \begin{array}{l} V_i \xrightarrow{\varphi_i} V_{i-1} \text{ injective} \\ \text{coker } \varphi_i \cong \mathcal{O}_{p_i}^{\oplus k_i} \end{array} \right\}$

defines $Gr_{\mathbb{P}_1^n}^{(k_1 \ldots k_n)} \to (\mathbb{P}^1)^n$

Fiber = product of Grassmannians by above argument (outside of diagonal...)

Assume $k_1 + \ldots + k_n = mr$, $r \in \mathbb{Z}$.

Then ${}_0Gr_{(\mathbb{P}^1)^n}^{(k_1 \ldots k_n)} := \{ (V_\bullet, \varphi_\bullet) \;/\; V_n \cong \mathcal{O}(-r)^{\oplus m} \}$ is an open subset.

- <u>The case $m=2$:</u> all $k_i = 1$, $n = 2r$;

then ${}_0Gr_{(p_1 \cdots p_n)}^{(1, \ldots, 1)} = \left\{ (A, V_\bullet) : \begin{array}{l} A \text{ is an } n \times n \text{ matrix of the form } \begin{bmatrix} * & I & & \\ \vdots & 0 & \ddots & \\ * & & \ddots 0 & I \end{bmatrix} \\ V_\bullet \text{ is a complete flag in } \mathbb{C}^n, AV_i \subset V_i, \\ A|(V_{i+1}/V_i) = p_{n-i} \end{array} \right\}$

$p_i \in \mathbb{A}^1$

If the p_i are distinct, then

$$ {}_0Gr^{(1\dots1)}_{(p_1\dots p_n)} \cong \left\{ A \,\middle/\, \begin{array}{l} \bullet \text{ eigenvals of } A \text{ are } p_1 \dots p_n \\ \bullet\ A \text{ is of the form } \begin{bmatrix} * & I & & \\ \vdots & & \ddots & \\ * & & & I \end{bmatrix} \end{array} \right\} $$

(<u>note</u>: in this case A determines the flag)

Seidel-Smith considered: ${}_0Gr^{(1\dots1)}_{\mathbb{C}^n \setminus \Delta} \to \mathbb{C}^n \setminus \Delta$ as a sympl. fibration, and its monodromy,

and defined a "braid group action on $DFuk\left({}_0Gr^{(1\dots1)}_{(p_1\dots p_n)}\right)$":

given $\underset{\text{Lagr.}}{L \subset} {}_0Gr^{(1\dots1)}_{(p_1\dots p_n)}$, $\quad \sigma \in B_n = \pi_1(\mathbb{C}^n \setminus \Delta)$

$\rightsquigarrow \sigma(L) \subset {}_0Gr^{(1\dots1)}_{(p_1\dots p_n)}$

(obtained by rescaled parallel transport along σ)

[used this to define a knot invariant, conj. = Khovanov homology]

- Case $\underline{n=2}$: $\quad \begin{array}{c} {}_0Gr^{(1,1)}_{\mathbb{P}^1\times\mathbb{P}^1} \\ \downarrow \\ \mathbb{P}^1\times\mathbb{P}^1 \end{array}$ fibers $\begin{cases} \text{above diagonal: } T^*\mathbb{P}^1 \\ \text{outside diagonal: } \\ \quad 2\times2 \text{ matrices w/ given eigenvals.} \\ \quad = \left\{ \begin{pmatrix} a & c \\ b & \tau - a \end{pmatrix} \middle/ \det = \delta \right\} \\ \quad = \text{conic in } \mathbb{C}^3 \end{cases}$

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joint work w/ S. Cautis:

We studied $DCoh\left({}_0Gr^{(1,\dots,1)}_{(0,\dots,0)}\right)$ & $DCoh\left(Gr^{(1\dots1)}_{(0,\dots,0)}\right)$

↑ diagonal case — opposite of Seidel-Smith!

$$ Y_n := Gr^{(1,\dots1)}_{(0,\dots,0)} = \left\{ \begin{array}{l} L_n \subset L_{n-1} \subset \dots \subset L_1 \subset L_0 = \mathbb{C}[z]^m \\ \text{st. } \dim L_i/L_{i+1} = 1 \text{ and } zL_i \subset L_{i+1} \end{array} \right\} $$

Let $Z_n^i = \{(L_\bullet, L'_\bullet) : L_j = L'_j \text{ for } j \neq i\} \subset Y_n \times Y_n \underset{p_2}{\overset{p_1}{\rightrightarrows}} Y_n$

- Case $m=2$:

Thm.: $\exists$ braid group action on $DCoh(Y_n)$ where $s_i \in B_n$ acts by

$$DCoh(Y_n) \longrightarrow DCoh(Y_n)$$
$$F \longmapsto p_{2*}(p_1^\alpha F \otimes \mathcal{O}_{Z_n^i})$$

This gives a knot invt which is equal to khovanov homology.

Note: $Z_n^i = \underset{L_i = L_i'}{\Delta} \cup X_n^i \times_{Y_{n-2}} X_n^i$

where $X_n^i = \{ L_\bullet : zL_{i-1} = L_{i+1} \}$
$\downarrow$ $\mathbb{P}^1$-bundle
Y_{n-2} (forget L_{i-1} and L_i)

Then $X_n^i \subset Y_n$, $X_n^i \downarrow Y_{n-2}$ $\Rightarrow$ (Kontsevich, Seidel-Thomas, Horja) spherical twist

$T_i = \text{cone}(F_i F_i^R \to id)$ where F_i is a spherical functor,

ie. $\begin{cases} F_i^R = F_i^L[-2] \\ F_i^R \circ F_i = id \oplus id[-2] \end{cases}$ $\quad \text{Hom}(F_i, F_i) = \mathbb{C}$, $\text{Ext}^j(F_i, F_i) = 0 \quad j = -1, -2.$

$\Rightarrow T_i$ is an autoequivalence.

- General case (m arbitrary)

Geometric state correspondence gives:

(1) $H_*\left(Gr^{(k_1 \ldots k_n)}_{(p_1 \ldots p_n)}\right) \cong \Lambda^{k_1}\mathbb{C}^m \otimes \ldots \otimes \Lambda^{k_n}\mathbb{C}^m$ $\leftarrow$ repⁿ of SL_m.

(2) $H_{mid}\left({}_0Gr^{(k_1 \ldots k_n)}_{(p_1 \ldots p_n)}\right) \cong \left(\Lambda^{k_1}\mathbb{C}^m \otimes \ldots \otimes \Lambda^{k_n}\mathbb{C}^m\right)^{SL_m}.$

Moreover, for p_i distinct, monodromy action of $S_n \equiv$ permutation of factors.

We expect that (1) can be categorified to $DCoh\left(Gr^{(k_1 \ldots k_n)}_{(0,\ldots,0)}\right)$

(2) can be categorified to $DFuk\left({}_0Gr^{(k_1\ldots k_n)}_{(p_1\ldots p_n)}\right)$, p_i distinct.

$\Rightarrow$ Want to get Br_n to act on these categories (thus giving knot invariants)

★ The case $k_i = 1, m-1$:

- On the symplectic side, Manolescu gave a B_n-action on $DFuk\left({}_0Gr^{(k_1\ldots k_n)}_{(p_1\ldots p_n)}\right)$ & defined a knot inv't (SL_m extension of Seidel-Smith).

- Sabin Cautis + J.K. $\Rightarrow$ braid group action on $DCoh\left(Gr^{(k_1\ldots k_n)}_{(0\ldots 0)}\right)$.

Take $n=2$ for simplicity:

$$Z^{(a,b)} = \{(L_\bullet, L'_\bullet) / L_2 = L'_2\} \subset Gr^{(a,b)} \times Gr^{(b,a)}$$

i.e.
$$\begin{tikzcd}[column sep=small] & \mathbb{C}[z]^m \arrow[dl, dash, "\text{codim. } a"'] \arrow[dr, dash, "\text{codim. } b"] & \\ L_1 \arrow[dr, dash, "b"'] & & L'_1 \arrow[dl, dash, "a"] \\ & L_2 = L'_2 & \end{tikzcd}$$

$$T: DCoh(Gr^{(a,b)}) \to DCoh(Gr^{(b,a)})$$
$$F \longmapsto p_{2*}(p_1^* F \otimes \mathcal{O}_Z)$$

<u>Theorem</u>: T gives an equivalence.
- if $a=b \in \{1, m-1\}$: this is again a spherical twist
- if $a=1$; $b=m-1$: then we have a Mukai flop. (Namikawa-Kawamata).

* <u>The case of arbitrary k_i:</u>

on the algebraic side, get $Z^{(a,b)} \subset Gr^{(a,b)} \times Gr^{(b,a)}$

for $a=b=2$, $m=4$: $Z^{(2,2)} \subset T^*G(2,4) \times T^*G(2,4)$

<u>stratified Mukai flop</u>

Namikawa showed that $p_{2*}(p_1^*(\cdot) \otimes \mathcal{O}_Z)$ is <u>not</u> an equivalence

Kawamata modified the construction to get an equivalence.

Work in progress will hopefully tell more ....