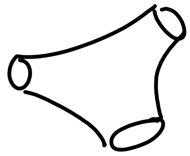
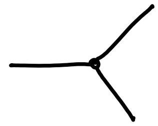
G. Mikhalkin . 26/01/08 - Phase-tropical curves



$\longrightarrow$
tropicalization



$\mathbb{C}$ curve   tropical curve

To go back, need $\underset{pt}{\circ} \to \underset{S^1}{\bigcirc}$ $\Rightarrow$ intermediate stage:

phase-tropical curve = trop. curve + system of phases

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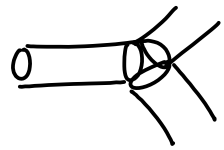


Applications: - enumerative geometry : criteria for realizability of trop. curve & multiplicities for realization over $\mathbb{C}$ and over $\mathbb{R}$

$\mathbb{C}$ ↙ phases $\in S^1$ $\mathbb{R}$ ↙ phases $\in \{+,-\}$

★ Algebraically: $\mathbb{T} = [-\infty, \infty)$ "+" = max , "·" = +

$K \xrightarrow{val} \mathbb{T}$ field w/ valuation, $val(a+b) \le \max(val\, a, val\, b)$.

Ex: K = Puiseux series $= \sum_{\lambda \in \mathbb{R}} a_\lambda t^\lambda$

Tropical curves in $\mathbb{R}^n$:

- Reminder: tropical curves in $\mathbb{R}^n$ = $\Gamma \subset \mathbb{R}^n$ proper embedding of a graph + integer weights on edges $w(E) \in \mathbb{N}_+$

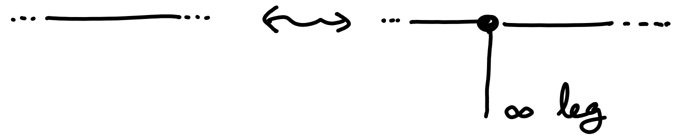
→ each edge E is straight, parallel to some primitive vector $v(E) \in \mathbb{Z}^n$

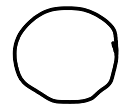
→ each vertex is balanced: $\sum_{E_j \text{ edge at vertex}} w(E_j)\, v(E_j) = 0$.

- Translation to abstract graph: Γ abstract graph, each oriented edge is prescribed a slope $s(E) \in \mathbb{Z}^n$ ($s(E) = w(E) v(E)$; $s(-E) = -s(E)$) st. at every vertex $\sum_{E_j} s(E_j) = 0$. (still miss: edge lengths!)

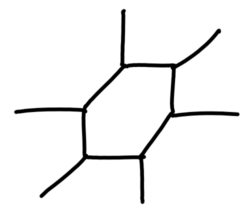
- Abstract tropical curve (w/out map to $\mathbb{R}^n$) is a metric graph, ie.
  - all edges except leaves are prescribed some finite length
  - leaves have ∞ length.

mod. tropical modifications =



<u>Ex</u>: any trop. elliptic curve  (modular param = length)

embeds in $\mathbb{R}^2$ as a trop. cubic after trop. modification (but length of cycle remains same)



<u>NB</u>: tropical lengths are measured wrt $GL_n(\mathbb{Z})$-distance : metric embeddings are via $GL_n(\mathbb{Z})$.

- A metric graph with a choice of slopes <u>may</u> define a tropical curve $h: \Gamma \to \mathbb{R}^n$ up to transl$^{\underline{n}}$

  if $g = b_1(\Gamma) > 0$, need to check that cycles close up properly ....

  The lengths need to satisfy g $\mathbb{R}^n$-valued linear equations: for each cycle of edges $E_1 \ldots E_k$ forming a cycle,

$$\sum_{i=1}^{k} s(E_i)\,\ell(E_i) = 0 \in \mathbb{R}^n$$

<u>Def</u>: A curve is <u>regular</u> if these $g \cdot n$ eq$^{\underline{ns}}$ are lin. independent, <u>superabundant</u> otherwise

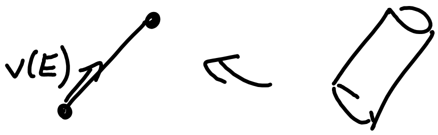
<u>NB</u>: this is related to deformation/obstruction theory.
Every <u>regular</u> trop curve arises as limit of amoebas of complex curves
Not always true for superabundant curves.

<u>Rmk</u>: - linear system is transverse $\Leftrightarrow$ dim. sol$^{\underline{ns}}$ = exp dim.
Hence regularity $\Leftrightarrow$ transversality $\Leftrightarrow$ dim $\mathcal{M}_{trop}$ = expected dim.

Phase-tropical curve in $\mathbb{R}^n$:

Think of each leg of a trop. curve as image (amoeba) of a holomorphic annulus in $(\mathbb{C}^*)^n$.

$$\mathbb{R}^n \xleftarrow{\text{Log}} (\mathbb{C}^*)^n = \mathbb{R}^n \times (S^1)^n \xrightarrow{\text{Arg}} (S^1)^n$$



edge E annulus $\longrightarrow$ geodesic of slope $v(E)$ in $(S^1)^n$.

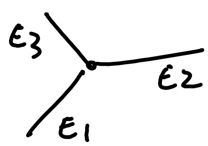
Set of such geodesics is parametrized by $(S^1)^n/v(E) \cong (S^1)^{n-1}$

$\rightarrow$ <u>phase of E</u>

<u>Hence:</u> phase-tropical curve := tropical curve + set of slopes $\sigma(E) \in (S^1)^n/v(E)$. st. compatibility at vertices

★ <u>Trivalent case</u>: every 3-valent vertex is planar

→ in $\mathbb{R}^2$-direction: (projection to 2-dim! subspace)



each $\sigma(E_i) \in (S^1)^2/v(E_i) \underset{\text{can.}}{\cong} \mathbb{R}/2\pi\mathbb{Z}$

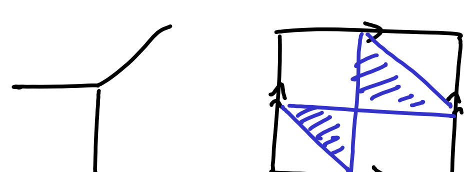
(with canonical orientation from orient. of $(S^1)^2$ & orient. of E_i)

$\Rightarrow$ <u>want</u>: $\sigma(E_1)+\sigma(E_2)+\sigma(E_3) \equiv m\pi \mod 2\pi$
where m = mult. of vertex $= |s(E_1) \wedge s(E_2)|$

(recall $s(E_i) = w(E_i)\, v(E_i)$)

→ in other $\mathbb{R}^{n-2}$ directions: $\sigma(E_1) = \sigma(E_2) = \sigma(E_3)$.

Compatibility comes from coamoeba:



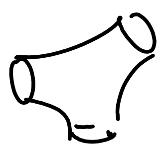
the 2 triangles have <u>equal</u> areas

★ For higher-valent vertices:

consider trivalent resolution of the vertex by inserting length 0 edges:

and equip length 0 edges with phases so that the trivalent vertices all satisfy the above condition.

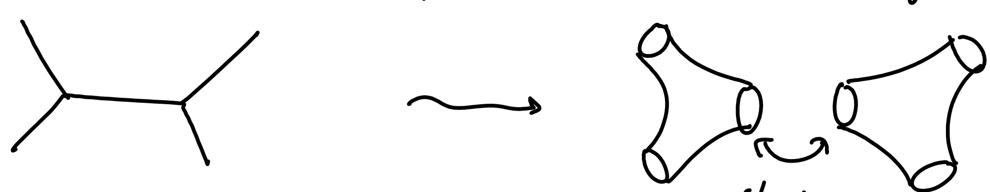
- For a trivalent tropical curve, each vertex has a $\mathbb{C}$-model



$\{z + w + 1 = 0\} \subset (\mathbb{C}^\times)^2$

3 boundary circles each give geodesic in $(S^1)^2$.

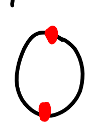
Phase structure on Γ := orientation-reversing isometry of the corresponding boundary circles for each non-leaf edge of Γ.

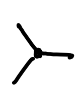


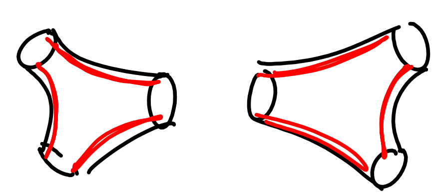
choose isometric (in $(S^1)^2$) identification of these circles (S^1-worth of such identif^{ns}).

- Real phase-tropical curves: each circle passes through a 2-torsion pt $\in (S^1)^n$

→ can mark 2-torsion points on circle ≡ real part of the curve:

on each circle 

and on the model for 



★ Each edge gluing must match the markings.

(⇒ 2 possible choices).