D. Orlov - 26/1/08 - III: B-branes in LG-models & MS

LG-models: X alg. var, smooth

(X, I, ω, B, W) W superpotential = regular f^{n} on X, nonconstant.

- Matrix fac^{ns}: $X = \operatorname{Spec} A$, $W \in A$ $\Rightarrow$

  - $DB_\lambda(X,W) = \left\{ P_1 \underset{p_0}{\overset{p_1}{\rightleftarrows}} P_0 \right\}$ P_1, P_0 projective finite dim modules

    $p_1 p_0 = (W-\lambda)\,Id$

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  - shift $[1]$: $P_0 \underset{-p_1}{\overset{-p_0}{\rightleftarrows}} P_1$; $[2] \simeq Id$

  - morphisms of MF's: in deg. 0 & 1 mod 2 & homotopies ...

$$\begin{tikzcd} \cdots & P_0 \arrow[r, shift left] \arrow[d, "f_0"'] & P_1 \arrow[l, shift left] \arrow[r, shift left] \arrow[d, "f_1"'] \arrow[dl, "g_0"'] & P_0 \arrow[l, shift left] \arrow[d, "f_0"] \arrow[dl, "g_1"'] & \cdots \\ \cdots & Q_0 \arrow[r, shift left] & Q_1 \arrow[l, shift left] \arrow[r, shift left] & Q_0 \arrow[l, shift left] & \cdots \end{tikzcd}$$

Def 1: $\| \; DB(Y,W) = \prod_{\lambda \in \mathbb{A}^1} DB_\lambda(X,W)$

Fact: - all DB_λ's are triangulated
- $DB_\lambda(X,W)$ is trivial if X_λ is smooth.

- Alternative def: $\|$ $D^b Coh(X_\lambda) \supset Perf(X_\lambda)$ ($\simeq$ if X_λ smooth)

  $\|$ $DB(Y,W) = \prod_{\lambda \in \mathbb{A}^1} \underbrace{D^b Coh(X_\lambda) / Perf(X_\lambda)}_{=: D_{sg}(X_\lambda)}$

Thm: $\|$ The 2 definitions are equivalent

- Note: If X carries an action of an alg. group G:

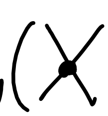
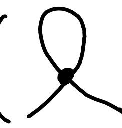
$$D^G_{sg} := D^b(Coh^G X) / Perf(X)^G$$

- More g$^{\text{ally}}$ A noetherian algebra $\Rightarrow$ can consider $D^b(A\text{-mod}) / Perf(A)$
(not necess comm.!!) $\qquad \overset{\shortparallel}{D^b(\text{Proj-}A\text{-mod})}$

- <u>Remark:</u> $\mathrm{Perf}\, X \simeq D^b\mathrm{Coh}\, X \iff X$ is regular (Rouquier)

- Let X separated noetherian scheme of finite Krull dim., s.t. for any coherent sheaf $\mathcal{F}$ there is a surjection from a vector bundle $\mathcal{E} \twoheadrightarrow \mathcal{F}$. (e.g. any quasiprojective X)

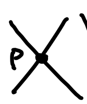
$\Rightarrow$ <u>Prop:</u> $j : U \hookrightarrow X$ Zariski open subset, s.t. $\mathrm{sing}(X) \subset U$. Then $j^* : D^b(\mathrm{Coh}\, X) \to D^b(\mathrm{Coh}\, U)$ (restriction) induces an <u>equivalence</u> $\bar{j}^* : D_{sg}(X) \xrightarrow{\sim} D_{sg}(U)$.

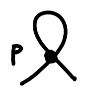
(<u>However</u> this isn't true in analytic topology: $D_{sg}\Big($  $\Big) \not\simeq D_{sg}\Big($  $\Big)$

<u>Def:</u> $\mathcal{T} \subset \overline{\mathcal{T}}$ idempotent completion (Karoubian envelope):
$\overline{\mathcal{T}}$ triangulated cat. with objects $A \xrightarrow{p} A$, $A \in \mathcal{T}$, $p^2 = p$
$\rightsquigarrow$ Coker p & Ker p.

$\rightsquigarrow$ get $D_{sg}(X) \subseteq \overline{D_{sg}(X)}$

<u>Thm</u> (Thomason): $\mathcal{T}$ essentially small tri. cat.: then there is a one-to-one correspondence between strictly full tri. subcat. $\mathcal{A} \subset \mathcal{T}$ st. $\overline{\mathcal{A}} = \overline{\mathcal{T}}$ and subgroups $H \subset K_0(\overline{\mathcal{T}})$.
In one direction, this is $\mathcal{A} \longmapsto \mathrm{Im}(K_0(\mathcal{A}) \to K_0(\overline{\mathcal{T}}))$

<u>NB:</u> e.g. $D_{sg}\Big($  $\Big)$ is idempotent complete; $\mathcal{O}_{E_1} \simeq \mathcal{O}_{E_2}[1]$ and their sum $\simeq \mathcal{O}_p$
$E_1 \quad E_2$

$D_{sg}\Big($  $\Big)$ isn't (doesn't have the formal summands $\mathcal{O}_{E_i}$)

So: $D_{sg}(X) \subseteq \overline{D_{sg}(X)}$, with K-group $H \subset K_0\big(\overline{D_{sg}(X)}\big)$

- $Z \subset X$ closed subscheme defined by ideal $\mathcal{I}$ $\Rightarrow$ <u>formal completion</u> $\widehat{X}_Z = \{Z, \varprojlim_n \mathcal{O}_X/\mathcal{I}^n\}$

<u>Def:</u> $\hat{X} := \widehat{X}_{sing(X)}$

<u>Thm:</u> X_1, X_2 two quasiproj. schemes, st. $\widehat{X_1} \cong \widehat{X_2}$ formal completions

Then $\overline{D_{sg}(X_1)} \cong \overline{D_{sg}(X_2)}$

<u>Idea proof:</u> $Z \overset{i}{\hookrightarrow} X$ $\Rightarrow$ let $D_Z(\mathrm{Coh}\, X) \subseteq D^b(\mathrm{Coh}\, X)$ all complexes whose cohomologies have support in Z

and let $\mathrm{Perf}_Z(X) = \mathrm{Perf}(X) \cap D_Z(\mathrm{Coh}\, X)$.

1) <u>Lemma:</u> $A \in D^b_Z(\mathrm{Coh}\, X)$ belongs to $\mathrm{Perf}_Z(X)$ iff $\forall B \in D^b_Z(\mathrm{Coh}\, X)$, $\mathrm{Hom}(A, B[i])$ are trivial for all but finitely many $i \in \mathbb{Z}$

2) <u>Lemma:</u> The natural embedding $D^b_Z(\mathrm{coh}\, X) \to D^b(\mathrm{Coh}\, X)$ induces a <u>full</u> embedding $D^b_Z(\mathrm{coh}\, X)/_{\mathrm{Perf}_Z} \to D_{sg}(X)$

3) <u>Prop:</u> Any object of $D_{sg}(X)$ is a direct summand of an object concentrated on $\mathrm{Sing}(X)$, ie. $\in D^b_{\mathrm{Sing}\, X}(\mathrm{Coh}\, X)/\mathrm{Perf}_{\mathrm{Sing}\, X} \subseteq D_{sg}(X)$.

Hence, for $Z = \mathrm{Sing}(X)$, $\overline{D_{sg}(X)} = \overline{D^b_Z(\mathrm{coh}\, X)/\mathrm{Perf}_Z}$

4) <u>Prop:</u> Let X_1, X_2 quasiproj. If $\widehat{X_1}_{Z_1} \simeq \widehat{X_2}_{Z_2}$ then $D^b_{Z_1}(\mathrm{coh}\, X_1) \cong D^b_{Z_2}(\mathrm{coh}\, X_2)$

Thus, by Lemma 1; can detect perfect complexes internally

$$\Rightarrow D^b_{Z_1}(\mathrm{coh}\, X_1)/\mathrm{Perf}_{Z_1} \cong D^b_{Z_2}(\mathrm{coh}\, X_2)/\mathrm{Perf}_{Z_2}$$

The thm then follows by Prop. 3.

- This argument gives us:

$$D^b_{\mathrm{sing}\,X}(\mathrm{Coh}\,X)/\mathrm{Perf}_{\mathrm{sing}\,X} \subseteq D_{sg}(X) \subseteq \overline{D_{sg}(X)}$$

- If X separated scheme of finite type then

$$\left\{\begin{array}{ll} D^b(\mathrm{Coh}\,X) & \text{has a strong generator} \\ D_{sg}(X) & \text{has a strong generator.} \end{array}\right.$$

$\underset{\text{Rouquier}}{\Rightarrow}$ $\exists$ DG-algebra A st. $\overline{D_{sg}(X)} \cong \mathrm{Perf}(A)$.

<u>Example:</u> $x \in X$ <u>isolated</u> sing. of X

k_x generator of $D_{sg}(X)$, $A := R\mathrm{Hom}_{D_{sg}(X)}(k_x, k_x)$

$$\Rightarrow \quad \begin{array}{ccccc} D_{sg}(X) & \subseteq & \overline{D_{sg}(X)} & \cong & \mathrm{Perf}(A) \\ \rotatebox{90}{$\subseteq$} & & & & \rotatebox{90}{$\subseteq$} \\ & & D^b_x(\mathrm{Coh}\,X)/\mathrm{Perf}_x & \cong & \underset{\substack{\text{triangulated} \\ \text{envelope}}}{\mathrm{Tri}(A)} \subset D(A) \end{array}$$

If eg. x is Gorenstein: $B := R\mathrm{Hom}_{\mathrm{Coh}\,X}(k_x, k_x)$

$$\Rightarrow \quad \overline{\mathrm{Perf}(B)/\mathrm{Perf}_{fd}(B)} \cong \overline{D^b_x(\mathrm{Coh}\,X)/\mathrm{Perf}_x(X)} = \overline{D_{sg}(X)}$$

finite dim! as k_x-module