D. Orlov – 23/1/08 – Generators & dimension of triangulated categories

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- **Def:** $\mathcal{E} \in \mathcal{T}$ tri cat. is a <u>classical generator</u> if the smallest tri-subcat. of $\mathcal{T}$ which contains $\mathcal{E}$ & closed under direct summands is $\mathcal{T}$ itself.

<u>Rmk:</u> if $\mathcal{E}$ classical generator and $\mathcal{T}$ <u>algebraic</u> then $\mathcal{T} = \mathrm{Perf}(A)$, A dg-algebra of $\mathrm{Hom}(\mathcal{E}, \mathcal{E}[i])$

- **Def:** $\mathcal{E} \in \mathcal{T}$ is called a <u>compact generator</u> if
  1) it is compact, ie. $\mathrm{Hom}(\mathcal{E}, \coprod_i X_i) \leftarrow \coprod_i \mathrm{Hom}(\mathcal{E}, X_i)$
  2) any X st. $\mathrm{Hom}(\mathcal{E}, X[i]) = 0 \ \forall i \Rightarrow X \simeq 0$.

<u>Rmk:</u> $\mathcal{T}$ algebraic & admits all coproducts, $\mathcal{E} \in \mathcal{T}$ compact generator $\Rightarrow \mathcal{T} = D(A)$

- Let X be a quasicompact & quasisep. scheme
  $\mathrm{Qcoh}(X) \subset \mathcal{O}_X\text{-mod}$.
  $D_{\mathrm{Qcoh}}(X) = \{$ derived cat. of $\mathcal{O}_X$-modules with Qcoh cohomologies $\}$

  <u>Rmk:</u> $D_{\mathrm{Qcoh}}(X) \not\simeq D(\mathrm{Qcoh}\, X)$ in general, but $\cong$ if X <u>separated</u>.

- **Def:** A complex of $\mathcal{O}_X$-modules is <u>perfect</u> if locally it is quasi-isom. to a bounded complex of vector bundles.
  ie. $P^\circ_{|U} \cong \{0 \to \mathcal{E}_i \to \dots \to \mathcal{E}_j \to 0\}$

  <u>Fact:</u> $\mathrm{Perf}(X) = \{$ compact objects of $D_{\mathrm{Qcoh}}(X)\} \subset D_{\mathrm{Qcoh}}(X)$.

<u>Thm</u> (Bondal - Van den Bergh):
X as above $\Rightarrow \exists\ \mathcal{E} \in \mathrm{Perf}(X)$ st. it classically generates $\mathrm{Perf}(X)$ and it compactly generates $D_{\mathrm{Qcoh}}(X)$.

<u>Corollary:</u> X as above $\Rightarrow \exists$ DG-alg. A st. $D_{\mathrm{Qcoh}}(X) \simeq D(A)$
$\mathrm{Perf}(X) \simeq \mathrm{Perf}(A)$

Let X quasiproj. scheme, $\mathcal{L}$ very ample line bundle $/X$

Thm: X q proj., $\dim X = d$, $\mathcal{L}$ very ample line bundle, $X \hookrightarrow \overline{X} \subset \mathbb{P}^N$, $\mathcal{L} = \mathcal{O}(1)_{|X}$

Then $\mathcal{E} = \bigoplus_{i=-d}^{0} \mathcal{L}^i$ is a classical generator for $\mathrm{Perf}(X)$.

Note: given $d+1$ sections of $\mathcal{L}$, $\underbrace{H_0 \dots H_d}_{= s_i^{-1}(0)}$, s.t. $\bigcap H_i = \emptyset$

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then $\left\{ (\mathcal{L}^{-1})^{\oplus(d+1)} \xrightarrow{s} \mathcal{O} \right\} \to \mathcal{L}^{-(d+1)} \to 0$

$\rightsquigarrow$ get all powers of $\mathcal{L}$, and from there, everything.

Strong generators:

- $\mathcal{T}$ tri. cat., $\mathcal{I}_1, \mathcal{I}_2$ full subcats. $\Rightarrow$ $\mathcal{I}_1 * \mathcal{I}_2 :=$ full subcat $\subset \mathcal{T}$ formed by all objects s.t. $\underset{\mathcal{I}_1}{M_1} \to M \to \underset{\mathcal{I}_2}{M_2} \to M_1[1] \to \dots$

- $\langle \mathcal{I} \rangle :=$ smallest full subcat. containing $\mathcal{I}$ and closed under direct sums, direct summands, and shifts.

- $\mathcal{I}_1 \diamond \mathcal{I}_2 := \langle \mathcal{I}_1 * \mathcal{I}_2 \rangle$

- Let $\mathcal{E} \in \mathcal{T}$, define $\langle \mathcal{E} \rangle_1 := \langle \mathcal{E} \rangle = \left\{ \mathcal{E}[n]^r, \bigoplus_{i=1}^{n} \mathcal{E}[n_i], \text{direct summands} \right\}$

$$\langle \mathcal{E} \rangle_i := \langle \mathcal{E} \rangle_{i-1} \diamond \langle \mathcal{E} \rangle_1$$

Def: $\mathcal{E}$ is called a <u>strong generator</u> of $\mathcal{T}$ if $\exists n \in \mathbb{N}$ st. $\langle \mathcal{E} \rangle_n = \mathcal{T}$.

Rmk: 1) $\mathcal{E} \in \mathcal{T}$ is a classical generator iff $\bigcup_i \langle \mathcal{E} \rangle_i = \mathcal{T}$.

2) if $\mathcal{T}$ has a strong generator then every classical generator is strong.

Indeed: if $\mathcal{T} = \langle \mathcal{F} \rangle_p$ strong,
$\mathcal{E} \in \mathcal{T}$ classical gen. $\Rightarrow \exists n$ st. $\mathcal{F} \in \langle \mathcal{E} \rangle_n$
$\Rightarrow$ then $\langle \mathcal{E} \rangle_{np} = \mathcal{T}$.

Def: dimension of tri. cat. $\mathcal{T}$:= smallest $d \in \mathbb{N}$ st. $\exists \mathcal{E} / \langle \mathcal{E} \rangle_{d+1} = \mathcal{T}$

Th (Bondal- Van den Bergh):

X smooth scheme $\Rightarrow$ $\mathrm{Perf}(X)$ has a strong generator.

Th (Rouquier)

X quasi-proj. $\Rightarrow$ the following are equivalent:
(1) X is regular
(2) $D^b(\mathrm{Coh}\, X) \cong \mathrm{Perf}\, X$
(3) $\mathrm{Perf}\, X$ has a strong generator.

Th (Rouquier)

X separated scheme of finite type over a perfect field k
$\Rightarrow D^b(\mathrm{Coh}\, X)$ has a strong generator. Thus $\dim D^b \mathrm{Coh}(X) < \infty$.

So: $\mathrm{Perf}\, X \subset \underset{\substack{\cup \\ \mathcal{E} \text{ strong generator}}}{D^b(\mathrm{Coh}\, X)} = \mathrm{Perf}(A)$, $H^*(A) = \mathrm{Hom}(\mathcal{E}, \mathcal{E}[i])$

Q: What is $\dim (D^b \mathrm{Coh}\, X)$?

Thm: Let X sep. reduced scheme of finite type. Then
$\dim D^b \mathrm{Coh}\, X \geq \dim X$

Thm (R.): Let X smooth q proj. scheme. Then
(1) $\dim X \leq \dim D^b \mathrm{Coh}\, X \leq 2 \dim X$
(2) if X affine then $\dim D^b \mathrm{Coh}\, X = \dim X$.

(☆) • **Thm.** C smooth curve $\Rightarrow \dim D^b \mathrm{Coh}\, C = 1$.

• $\mathbb{P}^n$: $\dim D^b \mathrm{Coh}(\mathbb{P}^n) = n$
$= \langle \mathcal{O}, \dots, \mathcal{O}(n) \rangle$

• **Def.** The spectrum I of $\mathcal{T}$:= $\{ n \in \mathbb{Z} \mid \exists \mathcal{E}, \langle \mathcal{E} \rangle_{n+1} = \mathcal{T}, \langle \mathcal{E} \rangle_n \neq \mathcal{T} \}$
Ex: for $\mathbb{P}^1$, $I = \{1, 2\}$.

Rmk: can use $\mathcal{O} \oplus \mathcal{O}(p)$ instead of $\mathcal{L}$ for generating D^b of a curve.

Q: $R\mathrm{Hom}(\mathcal{O} \oplus \omega, \mathcal{O} \oplus \omega) = ?$ on a curve of genus $g > 1$?
how to construct A_∞-structures?

Idea pf of (★):

1) Let $\mathcal{L}$ be a line bundle on C, $\deg \mathcal{L}$ large enough ($\geq 8g$)
$$\mathcal{E} = \mathcal{L}^{-1} \oplus \mathcal{O} \oplus \mathcal{L} \oplus \mathcal{L}^2.$$

2) $\mathcal{E} \in D^b(\mathrm{Coh}\, C)$, $\mathcal{E} \cong \bigoplus_i H^i(\mathcal{E})[-i]$

$\Rightarrow$ enough to show that any $\mathcal{F} \in \langle \mathcal{E} \rangle_2$.

Starting point: $\mathcal{F} = \underset{\text{torsion}}{T} \oplus \underset{\text{vect. bundle.}}{\mathcal{F}_0}$

- Lemma: For any torsion sheaf T, $\exists$ exact seq.
$$\underbrace{\{(\mathcal{L}^{-1})^{\oplus r_i} \to \mathcal{O}_C^{\oplus r_i}\}}_{\in\, \langle \mathcal{E} \rangle_2} \to T \to 0 \quad \Rightarrow T \in \langle \mathcal{E} \rangle_2.$$

- If $\mathcal{F}$ is a vect. bundle: HN-filtration
$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_n = \mathcal{F}, \quad \mathcal{F}_i/\mathcal{F}_{i-1} \text{ semistable}$$
$$\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) > \mu(\mathcal{F}_{i+1}/\mathcal{F}_i)$$
(where $\mu = c_1/\mathrm{rank}$).

Take i st. $\mu(\mathcal{F}_i/\mathcal{F}_{i-1}) \geq 4g > \mu(\mathcal{F}_{i+1}/\mathcal{F}_i)$.

Main lemma:
a) $(\mathcal{L}^{-1})^{\oplus r_1} \to \mathcal{O}^{\oplus r_0} \to \mathcal{F}_i \to 0$
b) $0 \to \mathcal{F}/\mathcal{F}_i \to \mathcal{L}^{\oplus s_0} \to (\mathcal{L}^2)^{\oplus s_1}$

Hence $\ker \alpha \to (\mathcal{L}^{-1})^{\oplus r_1} \xrightarrow{\alpha} \mathcal{O}^{\oplus r_0} \to \mathcal{F} \to \mathcal{L}^{\oplus s_0} \xrightarrow{\beta} (\mathcal{L}^2)^{\oplus s_1} \to \mathrm{Coker}\, \beta$

$$\begin{array}{ccc} \mathrm{Ext}^1(\mathcal{L}^{\oplus s_0}, \mathcal{O}^{\oplus r_0}) & \to & \mathrm{Ext}^1(\mathcal{F}/\mathcal{F}_i, \mathcal{F}_i) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ e' & \longmapsto & e \end{array}$$

Then $(\mathcal{L}^{-1})^{\oplus r_1} \oplus \mathcal{L}^{\oplus s_0}[-1] \xrightarrow[\varphi = \begin{pmatrix} \alpha & e' \\ 0 & \beta[-1] \end{pmatrix}]{} \mathcal{O}^{\oplus r_0} \oplus (\mathcal{L}^2)^{\oplus s_1}[-1]$

$\to \mathcal{F} \simeq \mathrm{Cone}(\varphi) \in \langle \mathcal{E} \rangle_2$. ◢