D. Orlov - 21/11/08 - Triangulated categories, DG categories, enhancements & generators

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- $\mathcal{M}$ triangulated cat. (e.g. $\mathcal{A}$ abelian category $\Rightarrow D(\mathcal{A})$ is a tri. cat.)

  $\mathcal{N} \subset \mathcal{M}$ closed under [1] and cones

  $\Rightarrow \mathcal{M}/\mathcal{N}$: $\quad ob(\mathcal{M}/\mathcal{N}) = ob(\mathcal{M})$

  $$\mathrm{Mor}_{\mathcal{M}/\mathcal{N}}(A,B) = \{ A \xleftarrow{s} A' \xrightarrow{f} B \}$$
  s.t. $\mathrm{Cone}(s) \in \mathcal{N}$

- DG-categories:

  - DG-algebra: $A = \oplus A_i$, $\quad d: A \to A$ of degree 1, $d^2 = 0$
  - DG-cat. $\mathcal{A}$ = k-linear category, st. $\forall$ objects X, Y,
    $\mathcal{A}(X,Y)$ are complexes of vector spaces / k
    $\mathcal{A}(Y,Z) \otimes \mathcal{A}(X,Y) \to \mathcal{A}(X,Z)$ Composition satisfies
    $$d(f \cdot g) = df \cdot g + (-1)^{\nu} f \cdot dg.$$

  - Graded hom. cat: $H^*(\mathcal{A})$, $\quad H^*(\mathcal{A})(X,Y) = \bigoplus_i H^i(\mathcal{A}(X,Y))$
    Homotopy cat: $H_0(\mathcal{A})$, $\quad H_0(\mathcal{A})(X,Y) = H^0(\mathcal{A}(X,Y))$

  - $C_{dg}(k) :=$ DG-cat. of all complexes of k-vect. spaces

  - Functors: $F: \mathcal{A} \to \mathcal{A}'$, $\quad F: ob\,\mathcal{A} \to ob\,\mathcal{A}'$
    $\forall X, Y$, $F(X,Y): \mathcal{A}(X,Y) \to \mathcal{A}'(FX, FY)$ morphism of complexes,
    compatible with composition

    $\mathrm{Hom}(\mathcal{A}, \mathcal{A}')$ forms a DG-category. (objs = functors
    mor = nat. transformations)

  - $F: \mathcal{A} \to \mathcal{A}'$ is called a <u>quasi-equivalence</u> if
    1) $F(X,Y): \mathcal{A}(X,Y) \to \mathcal{A}'(FX, FY)$ is a quasi-isomorphism
    2) $H_0(\mathcal{A}) \xrightarrow{\sim} H_0(\mathcal{A}')$ is an equivalence.

- <u>DG-modules</u> (right modules) over DG-cat:

$M: \mathcal{A}^{op} \to C_{dg}(k)$ DG-functor from opposite category

$\mathcal{A}^{op}(Y, X) = \mathcal{A}(X, Y)$

$\mathcal{A}^{op}$-DG-mod $= C_{dg}(\mathcal{A})$ right DG-modules

$X \in \mathcal{A} \mapsto X^\wedge = \mathcal{A}(-, X)$: $\mathcal{A} \hookrightarrow C_{dg}(\mathcal{A})$

Yoneda embedding; (image = all free, representable functors).

$\mathrm{Hom}_{C_{dg}(\mathcal{A})}(X^\wedge, Y^\wedge) = \mathrm{Hom}_{\mathcal{A}}(X, Y)$

- $\mathcal{A}$ DG-cat, then
  1) $\mathcal{H}(\mathcal{A}) := H_0(C_{dg}(\mathcal{A}))$
  2) $D(\mathcal{A}) := \mathcal{H}(\mathcal{A}) / Acycl(\mathcal{A})$

  where $N \in Acycl(\mathcal{A})$ if $H^*(N) = 0$.

<u>Prop:</u> $\mathcal{H}(\mathcal{A})$, $D(\mathcal{A})$ are triangulated categories

<u>Prop:</u> $\mathcal{H}(\mathcal{A}) \to D(\mathcal{A})$ has right and left adjoint functors.

- A DG-module F is called <u>semifree</u> if

$\exists$ filtration $0 = F_0 \subset F_1 \subset \dots \subset F_n \subset \dots \subset F$,

s.t. $F = \cup F_i$, and F_i / F_{i-1} are free DG modules

$=$ direct sums of $X^\wedge[n]$

$SF(\mathcal{A}) \subset C_{dg}(\mathcal{A})$ subcat. of semifree DG modules.

<u>Prop:</u>
1) $\forall$ DG-module M, $\exists F$ semifree s.t.

   $F \xrightarrow{\sim} M$ surjective quasi-equivalence

2) $\mathcal{H}(A)(F, N) = 0$ for any semifree F and acyclic N
3) $SF(\mathcal{A}) \subset C_{dg}(\mathcal{A})$ induces an equivalence

   $H_0(SF(\mathcal{A})) \simeq D(\mathcal{A})$.

- <u>h-projective</u> DG-modules $:= P$ s.t. $\mathcal{H}(\mathcal{A})(P,N)=0$ $\forall N$ acyclic.

Then $\begin{cases} SF(\mathcal{A}) \subset \mathcal{P}(\mathcal{A}) \subset C_{dg}(\mathcal{A}). \\ SF(\mathcal{A}) \subset \mathcal{P}(\mathcal{A}) \text{ is a quasi-equivalence} \\ H_0(SF(\mathcal{A})) \cong H_0(\mathcal{P}(\mathcal{A})) \cong \mathcal{D}(\mathcal{A}). \end{cases}$

- $SF_{fg}(\mathcal{A}) \subset SF(\mathcal{A}) := \{ 0 = F_0 \subset F_1 \subset \cdots \subset F_n = F \}$

s.t. F_p / F_{p-1} finite direct sums

f.g. semifree modules

Def: $SF_{fg} = \mathcal{A}^{pre\text{-}tr}$ DG-cat. of one-sided twisted complexes.

$\uparrow$ "pre-triangulated envelope" of $\mathcal{A}$

Def: triangulated cat. $Tri(\mathcal{A}) \subset \mathcal{D}(\mathcal{A}) :=$ the smallest triangulated subcat. of $\mathcal{D}(\mathcal{A})$ which contains all representable (free) modules $X^\wedge[n]$.

Then $Tri(\mathcal{A}) = H_0(\mathcal{A}^{pre\text{-}tr})$.

- Def: A DG-cat $\mathcal{A}$ is called <u>pretriangulated</u> if $\mathcal{A} \to SF_{fg} = \mathcal{A}^{pre\text{-}tr}$ is a quasiequivalence.

Then $H_0(\mathcal{A}) \xrightarrow{\sim} H_0(\mathcal{A}^{pre\text{-}tr}) = Tri(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$.

- $\mathcal{A}^{pre\text{-}tr}$, $C_{dg}(\mathcal{A})$, $SF(\mathcal{A})$, $\mathcal{P}(\mathcal{A})$ are pre-triangulated.

<u>Enhancements, and perfect objects:</u>

Def: An <u>enhancement</u> of a tri. cat. $\mathcal{T}$ is a pretriang. DG-cat. $\mathcal{A}$ together with an equivalence $\rho: H_0(\mathcal{A}) \xrightarrow{\sim} \mathcal{T}$.

Def.: Tri. cat. of *perfect objects* $Perf(\mathcal{A}) \subset \mathcal{D}(\mathcal{A})$:= closure of $Tri(\mathcal{A})$ under passage to direct summands in $\mathcal{D}(\mathcal{A})$.

i.e.: $M \oplus N \in Tri(\mathcal{A}) \Rightarrow M, N \in Perf(\mathcal{A})$

$\left(Perf(\mathcal{A}) = \overline{Tri(\mathcal{A})} \subset \mathcal{D}(\mathcal{A}) \right)$.

- $Perf_{dg}(\mathcal{A}) \subset C_{dg}(\mathcal{A})$, $H_0(Perf_{dg}(\mathcal{A})) = Perf(\mathcal{A})$.

Equivalent definition:

- for any tri. cat. T, an object $\mathcal{E} \in T$ is called *compact* if

$$\coprod_i Hom(\mathcal{E}, X_i) \xrightarrow[\text{isom.}]{\sim} Hom(\mathcal{E}, \coprod_i X_i) \quad \text{for any coproduct.}$$

Prop: An object $\mathcal{E} \in \mathcal{D}(\mathcal{A})$ is compact iff $\mathcal{E} \in Perf(\mathcal{A})$.

- **Thm:** If DG-cats. $\mathcal{A}$ and $\mathcal{B}$ are quasi-equivalent, then $\mathcal{A}^{pre\text{-}tr}$ & $\mathcal{B}^{pre\text{-}tr}$; $Perf_{dg}(\mathcal{A})$ & $Perf_{dg}(\mathcal{B})$; $C_{dg}(\mathcal{A})$ & $C_{dg}(\mathcal{B})$ are quasi-equivalent, and $Tri(\mathcal{A}) \cong Tri(\mathcal{B})$, $Perf(\mathcal{A}) \cong Perf(\mathcal{B})$, $D(\mathcal{A}) \cong D(\mathcal{B})$.

Algebraic triangulated cat. :=

$\underline{\mathcal{E}}$ stable cat. associated to a *Frobenius category* $\mathcal{E}$,

i.e. an exact category with enough projective & injective objects, and $Proj = Inj$.

stable cat. $\underline{\mathcal{E}}$:= same objects as $\mathcal{E}$, but morphisms are equiv. classes

$f: A \to B$ is $f \sim 0$ if it factors through an injective (& projective)

$$\begin{tikzcd} & JA \arrow[dr] & \\ f: A \arrow[ur] \arrow[rr] & & B \end{tikzcd}$$

- $\underline{\mathcal{E}}$ is triangulated.
- NB: subcats. & quotients of alg. cats. are algebraic.

$\mathcal{T}$ alg. triangulated cat., $\mathcal{G} \subset \mathcal{T}$ graded-full subcategory, i.e.

$$\mathcal{G}_{gr}(G, G') := \bigoplus_{n \in \mathbb{Z}} \mathcal{T}(G, G'[n])$$

$\Rightarrow$ <u>Th.</u>: (Keller) Let $\mathcal{T}$ alg. tr. cat., $\mathcal{G} \subset \mathcal{T}$ as above, then there exists a DG-cat. $\mathcal{A}$ st. $H^*(\mathcal{A}) = \mathcal{G}_{gr}$; and $F: \mathcal{T} \to D(\mathcal{A})$.

<u>Properties:</u>

a) F induces $\mathcal{T} \xrightarrow{\sim} \mathrm{Tri}(\mathcal{A})$ iff $\mathcal{T}$ coincides with the smallest full tri. subcat. of $\mathcal{T}$ containing $\mathcal{G}$.

b) F induces $\mathcal{T} \xrightarrow{\sim} \mathrm{Perf}(\mathcal{A})$ iff $\mathcal{T}$ is idempotent complete and coincides with the smallest tr. subcat. containing $\mathcal{G}$ and closed under direct summands.

c) $F: \mathcal{T} \hookrightarrow \mathcal{D}(\mathcal{A})$ is fully faithful (embedding) iff $\mathcal{G}$ forms a category of compact generators for $\mathcal{T}$, i.e. all objects of $\mathcal{G}$ are compact, and
$\mathcal{T}(G, X[n]) = 0 \;\forall G \in \mathcal{G} \;\forall n \in \mathbb{Z} \Rightarrow X = 0$

d) If in addition $\mathcal{T}$ admits all arbitrary coproducts then $F: \mathcal{T} \xrightarrow{\sim} \mathcal{D}(\mathcal{A})$.