D. Nadler - 25/1/08 - Part III

Recall our mirror symm. picture:

$$
\begin{tikzcd}[column sep=large]
 & \text{A-model} & \text{B-model} \\
 & G \arrow[r, leftrightarrow, "\text{Langlands}"] & G^\vee \\
\begin{array}{l}\text{4D TFT}\\ \text{Geom. Langlands (GL)}\end{array} & A_G \arrow[r, leftrightarrow, "\text{S-duality}"] \arrow[d, dashed, "\text{dim. red.}"'] & B_{G^\vee} \arrow[d, dashed] \\
\begin{array}{l}\text{“3D TFT”}\\ \text{character theory}\end{array} & Ch_G \arrow[r, leftrightarrow, dashed] & Ch_{G^\vee}
\end{tikzcd}
$$

Ch_G:

$S^1 \bigcirc \longmapsto$ D-mod$(G/G)_N$

D-modules on G mod adjoint action, supported in nilpotent cone

pt $\bullet \longmapsto$ monoidal category D-mod$(B\backslash G/B)$, whose center $\simeq$ the cat. assoc.d to S^1

4D TFT for G.L.:

2D reduction $\Longrightarrow$ C Riem. surface $\rightsquigarrow \mathcal{M}(C,G)$ Hitchin moduli space

then look at σ-model w/ maps $\Sigma \to \mathcal{M}(C,G)$.

Expect: $\mathcal{M}(C,G)$ $\mathcal{M}(C,G^\vee)$ are SYZ mirrors

$\mathcal{M}(C,G) \searrow \mathbb{A}^N \swarrow \mathcal{M}(C,G^\vee)$

(open subsets = dual torus fibrations over affine space)

Using the various interpretations (& various $\mathbb{C}$ structures):

- $\mathcal{M}(C,G) \simeq T^*\mathrm{Bun}_G(C)$ on A-side

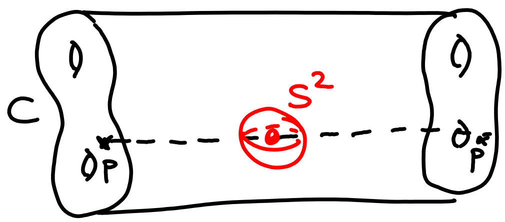
  Bun_G = moduli stack of holom. G-vector bundles on C
- $\mathcal{M}(C,G^\vee) \simeq \mathrm{Conn}_{G^\vee}(C)$ on B-side

  $\mathrm{Conn}_{G^\vee}$ = moduli of holom. vect. bundles + flat connections

So:

A		B
$A_G(C) = \text{D-mod}(\text{Bun}_G(C))$ $(\leftrightarrow \text{A-branes}(T^*\text{Bun}_G(C)))$	$\longleftrightarrow$	$B_{G^\vee}(C) = QC(\text{Conn}_{G^\vee}(C))$

- **3D part of story**: $p \in C \rightsquigarrow$ line operators give action $\mathcal{F}(S^2) \circlearrowright \mathcal{F}(C)$

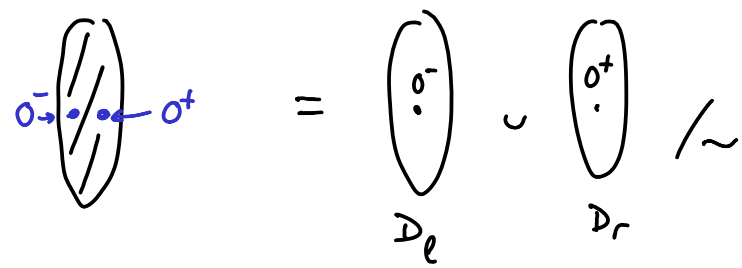


gives $\mathcal{F}(C) \otimes \mathcal{F}(S^2) \to \mathcal{F}(C)$. ✓

★ **Alg. geom. picture of S^2**: $(\rightsquigarrow$ what is $\mathcal{F}(S^2)$?)

S^2 = glue two discs along their boundary $\rightsquigarrow$ for us, glue all but a formal nbd of 0:

$\mathbb{D} =$



$$\Rightarrow A_G(\mathbb{D}) = \text{D-mod}(\text{Bun}_G(\mathbb{D})) = \text{D-mod}\left(L^+_{alg}G \backslash L_{alg}G / L^+_{alg}G\right)$$

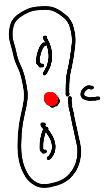
where alg. loop group $L_{alg}G = \text{Alg maps}(D^\times, G) = G((t))$

$$L^+_{alg}G = \text{Alg maps}(D, G) = G[[t]]$$

and $B_{G^\vee}(\mathbb{D}) = QC(\text{Conn}_{G^\vee}(\mathbb{D})) = BG^\vee \times_{BG^\vee} BG^\vee = BG^\vee$

$\Rightarrow$ **S-duality suggests**: expect $\text{D-mod}\left(L^+G \backslash LG / L^+G\right) \simeq QC(BG^\vee)$: *geometric Satake Corresp.*

Surface operators:

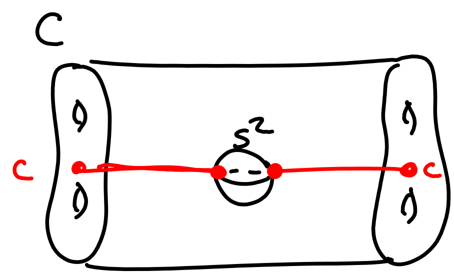


Fix $c \in C$ $\rightsquigarrow$

A: $\text{D-mod}(\widetilde{Bun}_G(C,c))$ ↙ holom. G-bundle over C + flag at c.

B: $QC(\widetilde{Conn}_{G^\vee}(C,c))$ ↙ $G^\vee$-conn. with a single pole at c + invariant flag at c.

Now:



$\Rightarrow$ $\mathcal{F}(c_\ell\, S^2\, c_r)$ acts on $\mathcal{F}(C,c)$.

↙ what is this?

A:

$\text{D-mod}({}_I\backslash LG/_I)$ $\longleftrightarrow$

$I \subset L^+G = \left\{ \begin{array}{l} \gamma : D \to G \text{ alg-map} \\ \gamma(0) \in B \end{array} \right\}$

(quotient only by those maps which preserve a flag in fiber at 0_ℓ, 0_r)

B:

$QC(St_{G^\vee})$

$St_{G^\vee} = \{ g \in G^\vee, B_\ell, B_r \; / \; g \in B_\ell \cap B_r \}$

g ↑ residue of flat conn. at each puncture

Thm (Bezrukavnikov): $\boxed{\text{D-mod}({}_I\backslash LG/_I) \simeq QC(St_{G^\vee})}$

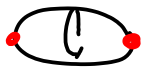
(NB: looking at K-groups of these categories $\rightsquigarrow$ Hecke algebras $\rightsquigarrow$ Kazhdan-Lusztig ...).

Dimensional reduction: $\mathcal{F}$ d-TFT $\longrightarrow$ $\mathcal{F}_{red}$ $(d-1)$-TFT

$$\mathcal{F}_{red}(M) := \mathcal{F}(M \times S^1)$$

More precisely: S^1 acts on $\mathcal{F}_{red}$ $\rightsquigarrow$ pass to S^1-invariant part of $\mathcal{F}(M \times S^1)$

4D



$\Big\downarrow$ S^1 reduction

$\underline{A}$		$\underline{B}$
$D\text{-mod}\left({}_{I}\backslash^{LG}/_{I}\right)$	$\simeq$	$QC(St_{G^\vee})$
$\Big\downarrow$ S^1-invt. part under rotation		$\Big\downarrow$ S^1-invt part under rotation
$D\text{-mod}\left({}_{B}\backslash^{G}/_{B}\right)$		$D\text{-mod}\left({}_{B^\vee}\backslash^{G^\vee}/_{B^\vee}\right)$
$\equiv Ch_G(pt)$ algebra		$\equiv Ch_{G^\vee}(pt)$ algebra

<u>Why?</u> • <u>A-side</u>: $\left({}_{I}\backslash^{LG}/_{I}\right)^{\mathbb{G}_m} = {}_{B}\backslash^{G}/_{B}$

Intuitively, $LG = \mathrm{Maps}(D^\times \to G)$: $\mathbb{G}_m$-invt part is G (constant loops).

This is "categorification of localization in equivt cohomology"

• <u>B-side</u>: what is the S^1-action on $St_{G^\vee}$?

<u>Prop</u>: $\Big\|$ $St_{G^\vee} = \mathcal{L}\left({}_{B^\vee}\backslash^{G^\vee}/_{B^\vee}\right)$ $\circlearrowleft$ loop rotation

(see last time)

<u>Local model</u>: $X = \operatorname{Spec} A \Rightarrow \mathcal{L}X = \operatorname{Spec} \Omega_A^{-\bullet}$

(using Hochschild-Kostant-Rosenberg) $\circlearrowleft S^1 \longleftrightarrow \circlearrowleft$ De Rham diff'l

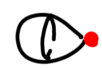
so $QC(\mathcal{L}X)^{S^1} = \Omega_A^{-\bullet}[d]\text{-mod}.$

$\xleftrightarrow[\text{duality}]{\text{Koszul}}$ $D\text{-mod}(X)$

✓

Work in progress

<u>4D</u>

$A_G\Big($  $\Big)$ $\overset{?}{\longleftarrow\!-\!-\!\longrightarrow}$ $B_{G^\vee}\Big($  $\Big)$

in progress

$\Big\downarrow$ S^1-invt $\qquad\qquad$ $\Big\downarrow$ S^1-invt

$\mathrm{Reps}(G) \longleftarrow - - - \longrightarrow \mathrm{Reps}(G^\vee)$

<u>Soergel's conj.</u>

<u>Strategy:</u> Langlands functoriality under base change: