D. Nadler - 24/11/08 - II

TFT : pt $\longrightarrow$ QCoh(G)

$\bigcirc$ G/G

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Problem: lack of smoothness : QC $\rightsquigarrow$ D-modules

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What are D-modules?

X smooth $\rightsquigarrow$ D-mod(X) = quasicoh. module on X
with action of $\underset{\text{vect. fields}}{Vect_X} \subset \underset{\text{diff. operators}}{Diff_X}$

- Note: D_X is filtered (order) with associated graded
$Gr\, D_X = \mathcal{O}_{T^*X}$ (symbol)

- quantization of Poisson bracket on T^*X
tells us how to go back from T^*X to D_X.

- Known: D-modules on X $\Longleftrightarrow$ A-branes on T^*X.

Singular support of a D-module = constr. set $\subset T^*X$, always coisotropic

Say a D-module is holonomic if support is Lagrangian,
i.e. has smallest dimension.

★ 2 examples from rep. theory:

1) Beilinson-Bernstein localization:

$\mathfrak{g}$ reducible, $X = G/B$ flag var.,

$\mathfrak{g} \longrightarrow Vect_X$

$U\mathfrak{g}/\text{center} = D(X)$

Thm (roughly): $U\mathfrak{g}$-mods $\leftrightarrow$ D_X-mods.

2) Lusztig's character sheaves:

$M \in Char_G = \mathcal{D}\text{-mod}(G/G)_{\mathcal{N}}$ ← supported in nilpotent cone $\subset \mathfrak{g}$.

ie: $T^*G \simeq G \times \mathfrak{g} \supset G \times \mathcal{N} \supset SS(M)$

__Back to TFT:__ $Ch_G :=$ character theory of G

"3D TFT" = $S^1 \longmapsto Ch_G(S^1) = Char_G$ E_2-category.

$pt \longmapsto ?$

__Guess:__ $G/G = \mathcal{L}(BG)$

→ expect: $Z(\mathcal{D}\text{-mod}(BG)) \overset{?}{=} \mathcal{D}\text{-mod}(G/G)$ → too small

NO: Morita fails!

$Z(\mathcal{D}\text{-mod}(G)) \underset{NO}{\overset{?}{=}} \mathcal{D}\text{-mod}(G/G)$ → too big

Recalling that ${}_B\backslash G/_B = BB \times_{BG} BB$

__Thm:__ $Z(\mathcal{D}\text{-mod}({}_B\backslash G/_B)) = Char_G$. → the right one

So: $pt \longmapsto \mathcal{D}\text{-mod}({}_B\backslash G/_B)\text{-mod}$.

- Favorite 1-brane in this theory?
  - (1) $Dmod(G/B) = \mathfrak{g}\text{-mod}$
  - (2) $Dmod({}_K\backslash G/_B) = G_{\mathbb{R}}\text{-reps}$.

__Next time:__ relation to mirror symmetry:

__main goal:__ the field theories $Ch_G \overset{?}{\simeq} Ch_{G^\vee}$ Langlands dual

__Implications would include:__ • $Ch_G(S^1) \simeq Ch_{G^\vee}(S^1)$

two categories of A-branes on $\neq$ spaces

- progress towards Soergel's conjecture on real groups:

Namely, group rep^ns come up as 1-branes!

$$G \longleftrightarrow G^\vee$$

$$\bigoplus_{\substack{G_{\mathbb{R}} \text{ real form} \\ \text{of } G}} D\text{-mod}\left(K\backslash G/B\right) \longleftrightarrow \bigoplus_{G^\vee_{\mathbb{R}}} D\text{-mod}\left(K^\vee\backslash G^\vee/B^\vee\right)$$