D. Nadler – 21/1/08 – Loop spaces in representation theory (w/ D. Ben-Zvi)

Q: What does repⁿ theory have to do with mirror symmetry?

1) Broad answer: aim of harmonic analysis:

$$\left\{\begin{matrix}\text{modules over}\\ \text{algebra of operators}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{modules over}\\ \text{a spectrum}\end{matrix}\right\}$$

simple modules $\longmapsto$ skyscraper sheaves at points.

Baby examples:

(1) $\mathbb{C}[t]$-modules $\xrightarrow{\text{[Jordan form]}}$ $QCoh(\mathbb{A}^1 = Spec\ \mathbb{C}[t])$

(2) Fourier analysis: T abelian group, loc. compact
$T^\vee = Hom(T, U(1))$

$$L^2(T) \xrightarrow{\sim} L^2(T^\vee)$$

operators = translation   points = eigenfunctions

Noncomm. case: G group, $\mu(G)$ = measures on G.

2) Precise answer: Geometric Langlands (D-modules etc....) from 4D gauge theory.

Plan: I) Today: Construction of (partial) 3D TFT Ch_G "character theory"

main object: $D\text{-mod}(G)$ (= A-branes on T^*G)

$\rightsquigarrow$ all repⁿ theory of G fits into Ch_G.

our focus: worldsheets $\underset{pt}{\bullet}$, $\bullet\!\!-\!\!\!-\!\!\bullet$, $\bigcirc_{S^1}$

II) Thursday: geom Langlands, 4D TFT, $A_G \underset{\text{S-duality}}{\simeq} B_{G^\vee}$

main object: $D\text{-mod}(LG)$

$\rightsquigarrow$ richer operator theory

III) Friday: S^1-equivariant dim! reduction

$$\begin{tikzcd} A_G \arrow[r, phantom, "\simeq"] \arrow[d, squiggly] & B_{G^\vee} \arrow[d, squiggly] \\ Ch_G \arrow[r, dashed, leftrightarrow] & Ch_{G^\vee} \end{tikzcd}$$

Implications: 1) $Ch_G(S^1) \simeq Ch_{G^\vee}(S^1)$

duality of Lusztig's char. sheaves

2) progress towards Soergel's conj.

identification of certain 1-branes in $Ch_G(pt)$ & $Ch_{G^\vee}(pt)$.

<u>Derived algebraic geom.</u> (Lurie; Toen-Vezzosi)

<u>Broad idea</u>: study <u>topological</u> (rather than discrete) comm. rings.

Our motivation: want <u>worldsheets</u> and <u>targets</u> of TFTs to live in same context.

$k = \mathbb{C}$, Alg_k = comm. k-algebras

look at functors

$$\begin{tikzcd} Alg_k \arrow[r, "\text{schemes}", "\text{spaces}"'] \arrow[dr, "\text{stacks}"'] & Sets \arrow[d, shift left, "\text{incl. (trivial groupoid)}"] \\ & Groupoids \arrow[u, shift left, "\pi_0"] \end{tikzcd}$$

or more generally stacks

<u>Examples</u>: 1) X scheme $\rightsquigarrow$ $X(A) := Hom(Spec\, A, X)$

functor of points

2) D-M stacks $X = Y$ scheme$/F$ finite group

give a functor A algebra $\rightsquigarrow$ $X(A)$ groupoid

3) Artin stacks $X = Y$ scheme$/G$ affine alg. group

e.g. $Y = pt \rightsquigarrow X = pt/G = BG$ $\longrightarrow$ top groupoids.

↳ classify in this way G-bundles over $Spec\, A$:

get Bun_G as a functor $Alg_k \to$ Top groupoids

Can go further — <u>higher</u> stacks :

$$\begin{tikzcd} \mathrm{Alg}_k \arrow[r, "\text{scheme}"] \arrow[dr, "\text{stack}"] \arrow[ddr, "\substack{\text{higher}\\ \text{stack}}"'] & \mathrm{Sets} \arrow[d, shift left] \\ & \mathrm{Groupoids} \arrow[u, shift left, "\pi_0"] \arrow[d, shift left] \\ & \mathrm{Spaces} \simeq \text{simplicial sets} \arrow[u, shift left, "\pi_1"] \end{tikzcd}$$

Ex: 4) $B(B\mathbb{G}_m)$ classifies gerbes.

And can actually generalize from Alg_k to

TopAlg_k = topological k-algebras $\simeq$ cdga_k

Commutative DG algs. $/k$

(homological, ie. nothing in deg > 0)

$$\begin{tikzcd} \mathrm{Alg}_k \arrow[r, "\text{schemes}"] \arrow[dr, "\text{stacks}"] \arrow[ddr, "\substack{\text{higher}\\ \text{stacks}}"'] \arrow[dd, shift left] & \mathrm{Sets} \arrow[d, shift left] \\ & \mathrm{Groupoids} \arrow[u, shift left, "\pi_0"] \arrow[d, shift left] \\ \substack{\mathrm{TopAlg}_k \\ \simeq \mathrm{cdga}_k} \arrow[uu, shift left, "H_0"] \arrow[r, "\substack{\text{DERIVED}\\ \text{STACK}}"'] & \mathrm{Spaces} \simeq \text{simplicial sets} \arrow[u, shift left, "\pi_1"] \end{tikzcd}$$

Ex: 5) "constant" spaces : $A \longmapsto X(A) = \mathfrak{X}$ top space (the same $\forall A$)

6) $X \searrow Y \swarrow Z$ affine $\Rightarrow$ $X \times_Y Z = \mathrm{Spec}\big(\underbrace{\mathcal{O}_X \otimes^L_{\mathcal{O}_Y} \mathcal{O}_Z}_{\text{cdga}}\big)$

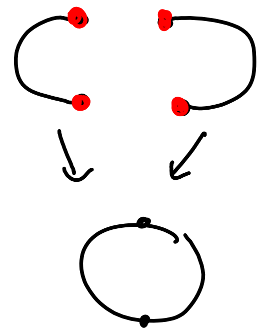
$\rightsquigarrow$ stacks = natural setting for allowing quotients

der. stacks = natural setting for allowing products.

<u>B-branes</u>: $\underset{\text{affine}}{X = \mathrm{colim}(U)} \longrightarrow \underset{\text{QCoh}}{QC(X)} = \lim_U QC(U)$.

<u>LOOP SPACES</u>: X scheme or Artin stack Y/G

<u>Def</u>: The topological <u>loop space</u> $\mathcal{L}X = \mathrm{Maps}(S^1, X)$.

<u>Alternative description</u>: $S^1 =$  affine cover

$\Rightarrow$ a map $S^1 \to X$ is two maps $I \to X$

s.t. restriction to $\mathrm{Map}(: \to X)$ $(= X \times X)$

agree with each other

Homotopically, $\mathrm{Map}(I, X) \sim X$ (constant maps)

and then restr. is $X \underset{\Delta}{\hookrightarrow} X \times X$

So: in derived sense,

$$\mathcal{L}X = \mathrm{Maps}(S^1, X) = X \underset{\Delta}{\times}{}^{R}_{X \times X}\underset{\Delta}{} X$$

derived product

<u>Ex</u>: (1) X smooth scheme $\rightsquigarrow$ $\mathcal{L}X = \mathrm{Spec}\left(\mathcal{O}_X \otimes^L_{\mathcal{O}_{X \times X}} \mathcal{O}_X\right)$

$$= \mathrm{Spec}(HH_*(\mathcal{O}_X))$$

Hochschild homology

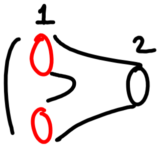
$$\underset{HKR}{=} \mathrm{Spec}(\Omega_X^{-\bullet})$$

(2) $X = BG = pt/G$ ...

B-branes on $\mathcal{L}X$: $QC(\mathcal{L}X)$ is an **E_2-category**

ie:

$$\begin{tikzcd} & QC\left(\mathrm{Maps}\left(\text{pants}, X\right)\right) \arrow[dr, "p_{2*}"] & \\ QC(\mathcal{L}X) \otimes QC(\mathcal{L}X) \arrow[ur, "p_1^*"] \arrow[rr, "\text{product}"'] & & QC(\mathcal{L}X) \end{tikzcd}$$



Suspect that $QC(\mathcal{L}X)$ is the <u>center</u> of some monoidal category

(<u>central structure</u> := choice of a canonical isom. between left & right tensoring

ie. $Z(\mathcal{C}) = \text{pairs } \{(c \in \mathcal{C}, \text{ isom. } -\otimes c \xrightarrow{\sim} c \otimes -)\}$

<u>NB:</u> $\mathcal{C}$ monoidal cat. then $Z(\mathcal{C}) \xrightarrow[\substack{E_1\text{-map} \\ (\text{monoidal map})}]{} \mathcal{C}$ center

and $Z(\mathcal{C})$ is an E_2-category

<u>Thm</u> (Ben-Zvi, Francis, N.):

For reasonable X, $Z(QC(X)) = QC(\mathcal{L}X)$

$$\begin{tikzcd} Z(QC(X)) \arrow[rr, equal] \arrow[dr] & & QC(\mathcal{L}X) \arrow[dl, dashed] \\ & QC(X) & \end{tikzcd}$$

<u>Reinterpreting:</u>
$$QC(\mathcal{L}X) = QC(X) \otimes_{QC(X\times X)} QC(X)$$
$$= \mathrm{End}_{QC(X\times X)}(QC(X))$$

• <u>Moreover:</u> (Morita-style statement)

given a $\underset{\text{"affine diagonal"}}{\underline{\text{reasonable}}}$ map $Y \to X$, get

$$\begin{tikzcd} Z(QC(Y \times_X Y)) \arrow[rr, equal] \arrow[dr] & & QC(\mathcal{L}X) \arrow[dl, "?"] \\ & QC(Y \times_X Y) & \end{tikzcd}$$

$? = p_{2*} p_1^*$

Namely: $\mathcal{L}X \longleftarrow \mathcal{L}X \times_X Y \longrightarrow Y \times_X Y$