D. Maulik - 25/1/08 - GW theory for families of K3 surfaces (w/ Pandharipande)

$\mathcal{X} = (X_t)$ 1-param. family of K3 surfaces, polarized - e.g. as quartics in $\mathbb{P}^3$.
$\downarrow \pi$
C

Study classical alg. geom., & enumerative problems, for the family?

E.g.: Noether-Lefschetz: for very general X_t, $\mathrm{Pic}(X_t) = \mathbb{Z}\langle \mathcal{O}(1)\rangle$

In codim. 1, $\mathrm{Pic}\, X_t$ jumps in rank - countably often, densely.

Q: How often does X_t jump?

ie: Fix Λ a rank 2 lattice, $\Lambda \supset (4)$, for how many t's is $\mathrm{Pic}(X_t) = \Lambda$?

Goal: Relate 3 theories:

1. Noether-Lefschetz theory of the family π: "enumerative theory of Hodge classes" $\rightsquigarrow$ Borcherds
2. $GW(\mathcal{X})$ in fiber curve classes $\rightsquigarrow$ Mirr. Symm. Yau-Zaslow
3. reduced GW-theory of a K3 surface $\rightsquigarrow$ Göttsche

cf. Harvey-Moore-Mariño - ... heterotic type II duality.

$$\text{"} GW(\mathcal{X}) = GW(K3) \times NL(\pi) \text{"}$$

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① $GW(\mathcal{X})$: Fix g, $\beta \in H_2(X)$ st. $\pi_*(\beta) = 0$,
then $\overline{M}_g(X,\beta)$ has virt. dim. 0

$$\Rightarrow \int_{[\overline{M}_g(X,\beta)]^{vir}} 1 = N^X_{g,\beta} \in \mathbb{Q}$$

$\xrightarrow{\text{Gopakumar-Vafa}}$ $n^X_{g,\beta}$, conjecturally $\in \mathbb{Z}$

<u>BPS count</u>: $\sum_{g,\beta} N^X_{g,\beta} \lambda^{2g-2} v^\beta := \sum_{g,\beta} n^X_{g,\beta} \lambda^{2g-2} \sum_d \frac{1}{d}\left(\frac{\sin d\lambda/2}{\lambda/2}\right)^{2g-2} v^{d\beta}$

$n^X_{g,\beta}$: GV-invariants

- $GW(X)$ is known in genus 0 when X is obtained from a complete intersection in a toric var. (Givental, Lian-Liu-Yau).

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② <u>GW(K3)</u>: $[\mathcal{M}_g(S,\gamma)]^{vir} = 0$ (virt. dim. $= g-1$ but the class itself is 0...!)

But: given $C \xrightarrow{f} S$, the obstruction space is

$$H^1(C, f^*T_S) \xrightarrow{\sim} H^1(C, f^*\Omega^1_S) \longrightarrow H^1(C,\omega_C) \simeq \mathbb{C}$$

using holom. symplectic form to get $T_S \xrightarrow{\sim} \Omega^1_S$; pullback of forms; $\simeq H^0(C,\mathcal{O}_C)^*$ by Serre duality

This gives us, over the moduli space of curves, a map

$$Ob_{\mathcal{M}} \xrightarrow{\varphi} \mathcal{O} \to 0$$

$\mathrm{Ker}(\varphi)$ defines a new obstruction theory on $\mathcal{M}$, with virt. dim. $= g$.

★ For $g=0$, this corresponds to a count of <u>nodal</u> rational curves $\rightsquigarrow$ Yau-Zaslow formula.

★ In g^{al}: $\mathbb{E} \to \mathcal{M}_g(S,\gamma)$ Hodge bundle, rank g, fiber $H^0(C,\omega_C)$ (pulled back from $\overline{\mathcal{M}_g}$).

$$\lambda_g = c_g(\mathbb{E}) \rightsquigarrow R_{g,\gamma} := \int_{[\mathcal{M}_g(S,\gamma)]^{vir}} (-1)^g \lambda_g \in \mathbb{Q}$$

Again define new invts by: for γ primitive,

$$\sum_{g,m} R_{g,m\gamma} \lambda^{2g-2} v^{m\gamma} = \sum_{g,m} r_{g,m\gamma} \lambda^{2g-2} \sum_d \frac{1}{d}\left(\frac{\sin d\lambda/2}{d\lambda/2}\right)^{2g-2} v^{dm\gamma}$$

Conj:
- $r_{g,\gamma} \in \mathbb{Z}$
- $r_{g,\gamma}$ only depends on $\langle \gamma, \gamma \rangle =: 2h-2$ $\rightsquigarrow r_{g,h}$

Known: for $g=0$: count of nodal rat'l curves

all g : Klemm, Katz, Vafa

$$\sum_h r_{0,h}\, q^{h-1} = \frac{1}{\Delta(q)} = \frac{1}{q \prod_n (1-q^n)^{24}}$$

(Yau-Zaslow, Bryan-Leung).

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③ NL theory: $\mathcal{M}_d$ = moduli space of polarized K3 surfaces of deg. d

$= \{(K3, L) \text{ st. } L^2 = d\}$

Fix Λ rank 2 lattice, containing (d).

$$D_\Lambda := \{ X \in \mathcal{M}_d \mid \mathrm{Pic}(X) \supset \Lambda \supset \langle L \rangle \} \quad \text{divisor in } \mathcal{M}_d$$

- A quasipolarized K3 family $\begin{array}{c}\mathcal{X}\\ \downarrow \pi \\ C\end{array}$ as considered above gives a map $C \to \mathcal{M}_d$.

$$\Rightarrow NL_\Lambda := \int_C i^* D_\Lambda \qquad \text{(\# members in family } (X_t) \text{ which lie in } D_\Lambda)$$

For $\Lambda = \begin{pmatrix} d & k \\ k & 2h-2 \end{pmatrix}$ call this $NL_\Lambda = NL_{h,k}$

NB: Count things with multiplicity = # manners in which $\Lambda \hookrightarrow \mathrm{Pic}(X)$, $e_1 \mapsto [L]$

e.g. for $k=0$, if $e_2 \mapsto \gamma$, could also do $e_2 \mapsto -\gamma$ $\Rightarrow$ mult. generically 2.

MAIN THM:

$$n^X_{g,k} = \sum_h r_{g,h}\, NL_{h,k}$$

by def^{n} $:= \sum_{\beta \mid \beta \cdot L = k} n^X_{g,\beta}$

IDEA PF:

- Thm (Borcherds):

> There is a vector-valued modular form $\vec{f} = (f_0(q), \dots, f_{d-1}(q))$ of weight $\frac{21}{2}$ and level 1, st. $NL_{h,k} = $ Coeff.t of $q^{\frac{k^2}{2d} - h + 1}$ in $f_{k \bmod d}(q)$.

ie: each $f_i(q)$ behaves well under action of $SL(2,\mathbb{Z})$

Namely: $\rho : SL_2(\mathbb{Z}) \to End(\mathbb{C}^d)$ rep^n $\Gamma(2d)$

then $\forall g \in SL_2$, $\vec{f}(g\cdot\tau) = (\dots)^{\dots}\rho(g)\vec{f}(\tau)$

- General result on Shimura varieties of type $O(2,n)$:

$\underline{n=2}$: Hirzebruch - Zagier
Gross-Zagier modular curves

$\Rightarrow$ with these ingredients, it's enough to prove $\underline{g=0}$ case, ie:

$$\sum_{k \equiv r \bmod d} n_{0,k}\, q^{k^2/2d} = \frac{1}{\Delta(q)} f_r(q)$$

Ex: quartic pencil of K3's in $\mathbb{P}^3$: [K-K-R-S.]

for $\begin{pmatrix} 4 & k \\ k & 2h-2 \end{pmatrix}$, $NL^{\pi}_{h,k} = $ coeff. of $q^{\frac{k^2}{8} - h + 1}$ in

$$\Theta(q) = \frac{1}{2^{22}}\left(3\,\theta_3^{21}(q^{1/4}) + \dots\right)$$

(involves θ_3 & θ_4 of $q^{1/4}$)

$$= -1 + 108\,q + 320\,q^{9/8} + \dots$$

$\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$: # nodal fibers (-2-class of deg. 0 !)

lines: $\begin{pmatrix} 4 & 1 \\ 1 & -2 \end{pmatrix}$ (-2-class of deg. 1)

- $NL(\pi)$ is independent of genus g

  $\Rightarrow$ once you know $GW_g(K3)$, get higher genus info on $\mathfrak{X}$

- Qualitative statement:

  $R_{g,\gamma}$, or more generally any invt in the reduced theory, is given by coeffs of a quasimodular form:

$$\sum_{\beta \text{ primitive}} \langle \dots \rangle \, q^{\langle \beta, \beta \rangle / 2} \quad \text{is a quasimodular form.}$$