C. Woodward – 23/1/08 – Functoriality for GW invts under quotients

(w/ Gonzalez, Mau, Ziltener)

- X smooth proj. $/\mathbb{C}$ $\longmapsto$ cohomological field theory (CohFT)

$$\mu^n : H^*(X,\mathbb{Q})^{\otimes n} \otimes H^*(\overline{M}_{0,n+1},\mathbb{Q}) \to H^*(X,\mathbb{Q})$$

(genus 0 GW invariants, dualizing one marked pt to make it an output)

satisfying splitting axiom.

Q: is this construction functorial?

- G compact, $G_{\mathbb{C}}$ Complexification, V rep^{n} of $G_{\mathbb{C}}$, $X \subseteq \mathbb{P}(V)$ smooth $G_{\mathbb{C}}$-variety.

$\to$ equivariant GW theory gives CohFT (Givental ...)

$$H^*_G(X)^{\otimes n} \otimes H^*(\overline{M}_{0,n+1}) \to H^*_G(X)$$

$X/\!\!/G = \Phi^{-1}(0)/G$ GIT/symplectic quotient, assume it's smooth.

($\Phi : X \to \mathfrak{g}^*$ moment map) $\qquad GW(X/\!\!/G)$ CohFT for $X/\!\!/G$.

Q: ‖ how are $GW_G(X)$ and $GW(X/\!\!/G)$ related?

* Mirror symm. motivation:

Hori–Vafa: conjectured relation b/w $GW((\mathbb{P}^{k-1})^n)$, $GW(Gr(k,n))$

proved by Bertram, Ciocan-Fontanine, Kim

who conjectured a relationship b/w $GW(X/\!\!/G)$, $GW(X/\!\!/T)^{\text{twisted}}$

Gauged GW theory:

Σ Riem. surf., $P \to \Sigma$ principal G-bundle

$\mathcal{A}(P) = \{\text{connections on } P\} \circlearrowleft \mathcal{G}(P)$ gauge transform^{ns}

moment map $= A \mapsto \mathrm{curv}(A) \in \Omega^2(\Sigma, \underbrace{(P \times \mathfrak{g})/G}_{= \mathrm{ad}\, P})$

Def: ‖ $\mathcal{A}(P,X) = \{ (A,u) \mid A \in \mathcal{A}(P),\ u \in \Gamma((P \times X)/G),\ \bar\partial_A u = 0 \}$ gauged holom. maps

- Sympl. structure on $\mathcal{A}(P, X)$:

$$(a_1, \xi_1), (a_2, \xi_2) \longmapsto \int_\Sigma Tr(a_1 \wedge a_2) + \varepsilon^{-1} Vol_\Sigma\, u^*\omega(\xi_1, \xi_2)$$

conn. 1-forms

vector field along u

- moment map for $\mathcal{G}(P)$-action on $\mathcal{A}(P,X)$:

$$(A, u) \longmapsto curv(A) + \varepsilon^{-1} \underbrace{vol_\Sigma}_{\Omega^2(\Sigma)}\ \underbrace{u^*\Phi}_{(P\times\mathfrak{g})/G} \in \Omega^2(\Sigma, ad P)$$

$$\mathcal{M}(P,X)_\varepsilon = \mathcal{A}(P,X) /\!/ \mathcal{G}(P) = \left\{ \begin{array}{l} curv(A) + \varepsilon^{-1} vol_\Sigma u^*\Phi = 0 \\ \bar\partial_A u = 0 \end{array} \right\}$$

<u>vortex eqns</u>

This has been studied by Cieliebak-Gaio-Salamon & by Mundet

- $\overline{M}(P, X)_\varepsilon$ = compactification by allowing bubbling in u.
- $\overline{M}(\Sigma, X)_\varepsilon = \bigcup_{\text{types } P} \overline{M}(P,X)_\varepsilon$ moduli of stable (nodal) vortices.

<u>Q</u>: dependence on ε?

<u>Answer</u>:
- as <u>$\varepsilon \to 0$</u>: Gaio-Salamon + Ziltener + ε

$$\Rightarrow \overline{M}_n(\Sigma, X)_\varepsilon \xrightarrow[\varepsilon\to 0]{} \overline{M}_{0,n+3}(X/\!/G)$$

n markings

+ vortices on $\Sigma = \mathbb{C}$
+ bubbles in X.

- as <u>$\varepsilon \to \infty$</u>: Gonzalez-Woodward for $\Sigma = \mathbb{P}^1$:

$$\overline{M}_n(\Sigma, X)_\varepsilon \xrightarrow[\varepsilon\to\infty]{} \overline{M}_{0,n+3}(X)/\!/G$$

$$\underset{\text{def}}{=} \left\{ u \text{ stable maps} \,\middle|\, \int_{\mathbb{P}^1} u^*\Phi = 0 \right\} / G$$

- <u>Vortex invariants</u>:

$$\begin{array}{ccc} & \overline{M}_n(\mathbb{P}^1, X)_\varepsilon & \\ \swarrow & & \searrow^{ev} \\ \overline{M}_{0,n+3} & & (X^n)_G \end{array}$$

$\rightsquigarrow$ in good cases, give a CohFT

$\rightsquigarrow$ <u>End result:</u> "Quantum non-abelian localization formula"

> $\exists$ diagram of CohFT's
>
> $$\begin{tikzcd} GW_G(X) \arrow[rr, "\text{morphism}", "\text{of CohFT's}"'] \arrow[dr, "\int_{\overline{M}_{0,n+3}(X)//G}"'] & & GW(X/\!/G) \arrow[dl, "\int_{\overline{M}_{0,n+3}(X/\!/G)}"] \\ & \mathbb{Q} & \end{tikzcd}$$
>
> which fails to commute by amount = an explicit sum of vortex invariants.

<u>NB:</u> compare with classical story:

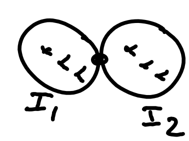
- $$\begin{tikzcd} K_G(X) \arrow[rr] \arrow[dr, "\int_{X\times G}"'] & & K(X/\!/G) \arrow[dl, "\int_{X/\!/G}"] \\ & \mathbb{Z} & \end{tikzcd}$$
  (invariant part of character)

  fails to commute by an explicit sum of fixed point contributions.

- $$\begin{tikzcd} H_G(X) \arrow[rr] \arrow[dr, "\int_{X\times g}"'] & & H(X/\!/G) \arrow[dl, "\int_{X/\!/G}"] \\ & \mathbb{Q} & \end{tikzcd}$$
  similarly.

<u>Morphisms of CohFT's:</u> (w/ Mau):

$$\partial \overline{M}_{0,n} = \bigcup_{I_1 \cup I_2 = \{1..n\}} D_{I_1 \cup I_2} \leftarrow$$



$$\begin{tikzcd} D_{I_1\cup I_2} \arrow[r, hook, "\eta_{I_1\cup I_2}", "\text{inclusion}"'] \arrow[d, "\varphi_{I_1\cup I_2}", "\text{separate the 2 components}"'] & \overline{M}_{0,n} \\ \overline{M}_{0,|I_1|+1} \times \overline{M}_{0,|I_2|+1} & \end{tikzcd}$$

<u>Notation:</u> for $\beta \in H^{\alpha}(\overline{M}_{0,n})$, let $\eta^{*}_{I_1\cup I_2}\beta = \sum_j \beta_{j,1} \otimes \beta_{j,2}$ Künneth components.

Def: A CohFT is a vector space V with maps

$$\mu^n : V^{\otimes n} \otimes H^*(\overline{M}_{0,n+1}) \to V \quad \text{s.t.}$$

$$\mu^n(\alpha_1, \dots, \alpha_n; \beta \wedge \delta_{I_1 \cup I_2}) = \sum_j \mu^{|I_2|}\left(\mu^{|I_1|}(\alpha_i (i \in I_1), \beta_{1,j}), \alpha_i (i \in I_2), \beta_{2,j}\right)$$

<u>splitting axiom</u>

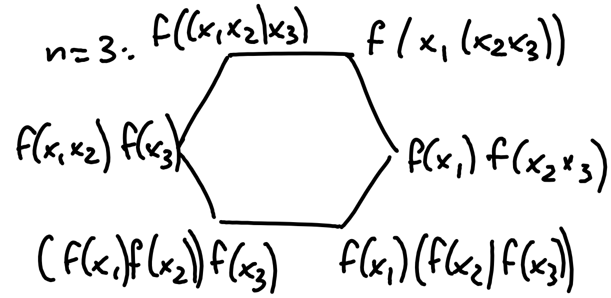
<u>Analogy:</u>
- A_∞-spaces = carry $\mu^n : X^n \times K_n \to X$ satisfying a splitting axiom

K_n = associahedron = moduli of stable disks
= metric ribbon trees
$= \overline{\{n \text{ distinct pts} \in \mathbb{R}\}} / \text{transl. dilation}$

$K_n \subset \overline{M}_{0,n+1}$ is the positive real locus

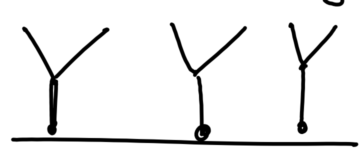
- Map of A_∞-spaces X, Y = collection $\{\phi^n : X^n \times K_{n,1} \to Y\}$ satisfying splitting axiom

$K_{n,1}$ = multiplihedron:
= as a CW-complex:



= colored metric rooted ribbon trees
(Boardmann-Vogt)

("mangroves")



length to any path is same

= quilted stable discs

$$= \overline{\left\{ \begin{array}{l} \text{scaled marked real lines, ie. lines } V \text{ with} \\ \text{transl.-invariant volume form \& markings} \\ \qquad TV \simeq V \times k \end{array} \right\}}$$

(ie.: "scaled" means we only quotient by translations, not by dilation)

The latter is best for us: we set

$$\overline{M}_{0,n+1,1} = \overline{\{\text{scaled marked complex lines}\}}$$

$K_{n,1} \subset \overline{M}_{0,n+1,1}$ real positive locus

projective var. w/ toric sings.

- $$\partial \overline{M}_{0,n+1,1} = \bigcup_{I \subset \{1..n\}} D_I \cup \bigcup_{I_1 \cup \dots \cup I_r = \{1..n\}} D_{I_1 \cup \dots \cup I_r}$$

<u>Def:</u> $(V, \mu_V^\bullet)$, $(W, \mu_W^\bullet)$ CohFT's: a <u>morphism</u> of CohFT's is a collection $\phi^n : V^{\otimes n} \otimes H^*(\overline{M}_{0,n+1,1}) \to W$ satisfying splitting axiom for Cartier divisors

- $$\overline{M}_{n+1,1}(\mathbb{C}, X) = \overline{\left\{ \begin{array}{c} \text{finite energy vortices on } \mathbb{C} \\ \text{with values in } X \end{array} \right\}} \Big/ \text{iso}$$

$$\begin{tikzcd} & \overline{M}_{n+1,1}(\mathbb{C}, X) \arrow[dl, "f"'] \arrow[dr] & \\ \overline{M}_{0,n+1,1} & & X_G^n \times X/\!\!/G \end{tikzcd}$$

<u>Conj:</u> gives a morphism of CohFT's $GW_G(X) \to GW(X/\!\!/G)$.