A. Kuznetsov – 26/1/08 – Derived categories of Fano 3-folds

X^n smooth, Fano $\rightsquigarrow$ $(\mathrm{Pic}\, X, \bullet, K_X)$
(intersection form)

Consider mostly case $\mathrm{Pic}(X) = \mathbb{Z}H$
polarization

Then "basic invts": $\begin{cases} \bullet \text{ degree: } d_X = H^n \\ \bullet \text{ index: } i_X \text{ st. } K_X = -i_X H \end{cases}$

Thm: X Fano mfld $\Rightarrow$ $i_X \leq n+1$. Moreover,
if $i_X = n+1$ $\Rightarrow$ $X \cong \mathbb{P}^n$
$i_X = n$ $\Rightarrow$ $X \cong Q \subset \mathbb{P}^{n+1}$
quadric

- If $n=3$: → $i_X = 4$: $\mathbb{P}^3$
→ $i_X = 3$: $Q. \subset \mathbb{P}^4$
→ $i_X = 2$: then $d_X \leq 5$:
  - $d=5$: $Y_d = Gr(2,5) \cap \mathbb{P}^6$
  - $d=4$: $Y_d = Q_1 \cap Q_2 \subset \mathbb{P}^5$ quadrics
  - $d=3$: $Y_d \subset \mathbb{P}^4$ cubic 3-fold
  - $d=2$: $Y_d \xrightarrow[2:1]{} \mathbb{P}^3$ ramified at a deg. 4 surface
  - $d=1$: $Y_d \underset{\text{deg. } 6}{\subset} \mathbb{P}(3,2,1,1,1)$

→ $i_X = 1$: then a general hyperplane section $S \subset X$ is a K3 surface of degree $d = 2g-2$,
$2 \leq g \leq 12$, $g \neq 11$.
  - $g=12$: $X_{22} \subset Gr(3,7)$ zero locus of a sec. of $(\Lambda^2 U^*)^{\oplus 3}$
  - $g=10$: $X_{18} = G_2Gr(2,7) \cap \mathbb{P}^{10}$ isotropic G_2 grassmanian
  - $g=9$: $X_{16} = SGr(3,6) \cap \mathbb{P}^{10}$ isotropic sympl. grassm.
  - $g=8$: $X_{14} = Gr(2,6) \cap \mathbb{P}^9$

- $g=7$: $X_{12} = OGr_+(5,10) \cap \mathbb{P}^8$
- $g=6$: $X_{10} = Gr(2,5) \cap \mathbb{P}^7 \cap Q_6$
- $g=5$. $X_8 = Q_1 \cap Q_2 \cap Q_3 \subset \mathbb{P}^6$
  3 quadrics
- $g=4$: $X_6 = Q \cap F_3 \subset \mathbb{P}^5$
  quadric $\cap$ cubic
- $g=3$: $X_4 = F_4 \subset \mathbb{P}^4$ quartic 3-fold
- $g=2$: $X_2 \xrightarrow{2:1} \mathbb{P}^3$ ramified at deg. 6 surface

<u>Mukai approach</u> to this classification:

e.g. case $i_X = 1$: $S \subset X$ hyp. section = K3 surf. of genus g.
$\mathrm{Pic}(S) = \mathbb{Z} \cdot H$.

- if $g = ab \rightsquigarrow \mathcal{M}od_S(v = a + H + b) = \{\text{point}\}$

moduli of vector bundles /S
with $\begin{cases} \text{rank } r = a \\ c_1 = H \\ \chi = a+b \end{cases}$ (by Mukai's classif^n of vect. bundles on K3's)

$E \to S$ the unique such bundle

Can check: E extends to X, and is gen^d by global sections
$H^0(E)$ has rank $a+b$.; at each pt, those that vanish = codim. a subspace.

Hence get a map $X \to Gr(a, a+b)$
induced by sections of E

Use this for the classification

using simult^ly. all decomp. $g = ab$ to get several maps
(e.g. $a=1, g=b$ gives $X \to \mathbb{P}^{g+1}$)

Derived cat's & semiorthogonal decomp^{ns}:

Def: A semiorthogonal decomp. (s.o.d.) of a tri. cat. $\mathcal{T}$ is a sequence $\mathcal{A}_1, \dots, \mathcal{A}_n$ of full tri. subcats. st.

1) $\mathrm{Hom}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $i > j$

2) $\forall T \in \mathcal{T}$ $\exists$

$$\begin{tikzcd}
0 = T_n \arrow[r] & T_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & T_2 \arrow[r] & T_1 \arrow[r] \arrow[d] & T_0 = T \arrow[d] \\
& A_n \in \mathcal{A}_n \arrow[ul, dashed] & & & A_2 \in \mathcal{A}_2 \arrow[ul, dashed] & A_1 \in \mathcal{A}_1 \arrow[ul, dashed]
\end{tikzcd}$$

Write $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_n \rangle$

Lemma: Assume $\alpha : \mathcal{A} \hookrightarrow \mathcal{T}$ fully faithful functor st. $\exists$ right adjoint $\alpha^! : \mathcal{T} \to \mathcal{A}$. Then $\mathcal{T} = \langle \mathcal{A}^\perp, \mathcal{A} \rangle$ where $\mathcal{A}^\perp = \{ T \in \mathcal{T} \,/\, \mathrm{Hom}(\mathcal{A}, T) = 0 \} = \ker(\alpha^!)$

Pf:
- adjunction $\Rightarrow$ $\exists$ natural transf. $\alpha\alpha^! \to \mathrm{Id}$

  So: $\forall T \in \mathcal{T}$, $\underbrace{\alpha\alpha^! T}_{\in \alpha(\mathcal{A})} \to T \to \underbrace{T' := \text{Cone}}_{\text{want: } \in \mathcal{A}^\perp \,?}$.

- Note that α fully faithful $\Rightarrow$ $\alpha^!\alpha \cong \mathrm{Id}$

  Applying $\alpha^!$: $\alpha^!\alpha\alpha^! T \xrightarrow[\cong]{} \alpha^! T \to \alpha^! T'$

  so $\alpha^! T' = 0$, ie. $T' \in \mathcal{A}^\perp$.

  gives desired decomp^n of $T \in \mathcal{T}$ into $\mathcal{A}$ & $\mathcal{A}^\perp$ ◢

Ex: $\mathcal{A} = D^b(\mathrm{Vect}_k)$, $\alpha : \mathcal{A} \to \mathcal{T}$, $k \mapsto E$

α is fully faithful $\Leftrightarrow$ E is an exceptional object.

The right adjoint is $\alpha^! = \mathrm{Hom}(E, -)$

Let X Fano mfld, $\dim X = n$, $i_X = i$.

$\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(i-1)$ form an exc. collection.

$\Rightarrow D^b(X) = \langle \mathcal{B}_X, \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(i-1) \rangle$, where by construction

$$\mathcal{B}_X = \{ F \in D^b(X) \,/\, H^\bullet(F) = H^\bullet(F(-1)) = \dots = H^\bullet(F(1-i)) = 0 \}$$

- $X = \mathbb{P}^n \Rightarrow \mathcal{B}_X = 0$: $D^b(\mathbb{P}^n) = \langle \mathcal{O}, \dots, \mathcal{O}(n) \rangle$
- $X = Q^n \subset \mathbb{P}^{n+1}$ $\Rightarrow$ $\mathcal{B}_X$ nontrivial, has 1 or 2 exc. objects depending on parity of n
  Kapranov
  in case $n=3$, $\mathcal{B}_X = D^b(\underbrace{\bullet \ \bullet}_{\text{2 points.}})$
  (orthogonal to each other)

- if $\dim X = 3$, $i_X = 2$: $D^b(Y_d) = \langle \mathcal{B}_d, \mathcal{O}, \mathcal{O}(1) \rangle$
- if $\dim X = 3$, $i_X = 1$, g even : $D^b(X_{2g-2}) = \langle \mathcal{A}_g, \mathcal{O}, E \rangle$
  $g = 2.t(\mathcal{O}, E)$
  E = extension of $E \to S$ constructed as above.

<u>Conjecture:</u> $\mathcal{B}_d \cong \mathcal{A}_{2d+2}$ for $d = 1, 2, 3, 4, 5$.

More precisely: we have maps $\underbrace{\mathcal{M}(Y_d)}_{\text{moduli of Fano}} \to \underbrace{\mathcal{M}(\mathcal{B}_d)}_{\text{moduli of category.}}$

$$\mathcal{M}(X_{2g-2}) \to \mathcal{M}(\mathcal{A}_{2d+2}).$$

<u>Rmk:</u> $Y_5 = Gr(2,5) \cap \mathbb{P}^6$ is rigid $\Rightarrow$ only one $\mathcal{B}_5$
X_{22} has deform^{ns}, but $\mathcal{A}_{12}$ remains the same.

$\star$ We have essentially proved the conj for $d = 3, 4, 5$.

- If $d = 5$: $\mathcal{B}_5 \cong D^b(\bullet \substack{\to \\ \to \\ \to} \bullet) \cong \mathcal{A}_{12}$.

• **If $d=4$:** $\mathcal{B}_4 \cong D^b(C_2$  $) \cong \mathcal{A}_{10}$

hyperelliptic genus 2 curve

In fact: recall $Y_4 = Q_1 \cap Q_2 \subset \mathbb{P}^5$; can relate Y_4 to C_2:

- Y_4 = moduli of bundles$/C_2$

<u>or</u>: • look at pencil of quadrics $\subset \mathbb{P}^5$ gen^d by Q_1 and Q_2, 6 of them are singular, curve $\Longleftrightarrow$ singular locus.

And in fact $X_{18} = G_2 Gr(2,7) \cap \mathbb{P}^{10} \subset \mathbb{P}^{13}$

- look at pencil of hyperplane sections containing the $\mathbb{P}^{10}$ & singular locus of this family of 4-folds $\rightsquigarrow C_2$ 

So correspondence is geometric on both sides.

• **If $d=3$:** $\mathcal{B}_3 \cong \mathcal{A}_8$

<u>Rmk</u>: this is a "fractional CY cat.": $S^3 \simeq [5]$

"CY dim. $\frac{5}{3}$"

<u>Rmk</u>: more geometrically there seems to be a relation b/w moduli spaces of instanton bundles over X_g and Y_d.