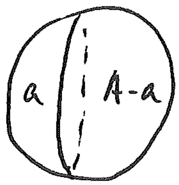
$\mathcal{F}_w(\mathbb{CP}^1) = \{$ weakly unobstructed Lag'ns st $\mu^0 = w.1\}$

is an honest $A_\infty$- categ. for all $w \in \mathbb{K}$.




$(L, \nabla)$

$$\mu^0 = W(z).\,\mathrm{id}$$
$$W(z) = z + \frac{T^A}{z}$$

Have an object of $\mathcal{F}_{W(z)}$, but get $HF(L,L) = 0$ unless $L$ is an equator: $a = \frac{A}{2}$. (otherwise, $L$ is displaceable !)

Also must have hol $\pm 1$.

So, non-trivial objs $(HF(L,L) \neq 0)$ are at $z = \pm T^{A/2}$.

$\rightsquigarrow W = z + \frac{T^A}{z} = \pm 2T^{A/2}$.

These are the critical pts of $W$.

Mirror: $MF_w(X^\vee, W)$ matrix factorizations.

Assume: $X^\vee = \mathrm{Spec}\, R$ affine.

$R^{\oplus k} \underset{\partial}{\overset{\partial}{\rightleftarrows}} R^{\oplus k}$, $\boxed{\partial^2 = (W-w).\,\mathrm{id}}$ matrix factorization

$MF(\mathbb{K}^*, z + \frac{T^A}{z} - w)$ non-trivial iff $w = \pm 2T^{A/2}$.

$\mathbb{K}[z^{\pm 1}] \underset{g}{\overset{f}{\rightleftarrows}} \mathbb{K}[z^{\pm 1}]$ $\quad f \cdot g = W - w$

$$z + \frac{T^A}{z} \pm 2T^{A/2} = (z - T^{A/2})\left(1 - \frac{T^{A/2}}{z}\right)$$

This non-trivial matrix fact is mirror to equator w/ hol $= \pm 1$.

Orlov: $MF(W-w) \simeq D^b\mathrm{Sing}(W-w) := D^b\mathrm{Coh}(W^{-1}(w)) / \mathrm{Perf}$

11/3

<u>Recap</u>: $X$ Kähler $\supset D$ divisor (reduced, normal crossings), $|D| = -K_X$.

$X^\circ = X \setminus D$ is open CY

<u>Candidate mirrors</u>:

$X^\vee$ mirror to $X^\circ$ : "moduli space" of $T^n$-objects of $\mathcal{F}(X^\circ)$ (eg: Lagr tori w/ rank 1 local systs)

* <u>Family Floer theory</u> (Abouzaid, Fukaya) gives a more robust approach (eventually). So far: Lagr torus fibs w/o singularities, Fukaya: simple singularities in dim 4.

* <u>SYZ</u>: input = Lagr $T^n$-fibr on $X^\circ$, w/ a Lagr. section.

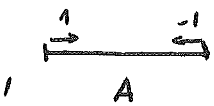
$W \in \mathcal{O}(X^\vee)$ measures how Lagrs become weakly unobstructed objects in $\mathcal{F}(X)$ : $\mu^0 = W \cdot \mathrm{id}$.

($W$ is a weighted count of holo. disks w/ $[u]\cdot D = 1$)

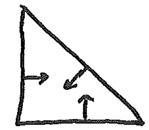
<u>Ex</u>: $(\mathbb{C}^*)^n \longleftrightarrow (\mathbb{C}^*)^n$ or $(\mathbb{K}^*)^n$

If $X$ toric Fano, $D$ = toric divisors, then

$X \leftrightarrow X^\vee = (\mathbb{C}^*)^n$, $W$ = Laurent poly whose terms are given by | facets of polytope
| rays of fan

Ex: $\mathbb{CP}^1 \longleftrightarrow \mathbb{C}^*$, $W = z + \frac{T^A}{z}$, $\int_{\mathbb{CP}^1}\omega = A$, 

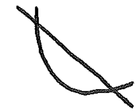


$\mathbb{CP}^2 \longleftrightarrow (\mathbb{C}^*)^2$, $W = x + y + \frac{T^A}{xy}$ 

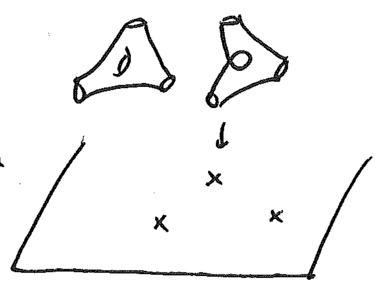
$D =$ 

Ex (non-toric):

$\mathbb{C}^2 \setminus \{uv=1\} \longleftrightarrow \mathbb{C}^2 \setminus \{uv=1\}$

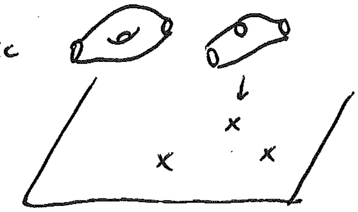
$(\mathbb{CP}^2, D =$  $) \longleftrightarrow \mathbb{C}^2 \setminus \{uv=1\}$, $W = u + \frac{T^A v^2}{uv-1}$

This mirror to $\mathbb{CP}^2$ is equivalent to toric mirror above!:

toric mirror 

$$(\mathbb{C}^*)^2 \xrightarrow{} \mathbb{C} \quad x+y+\frac{T^A}{xy}$$

crit vals $= 3\sqrt[3]{1}\, T^{A/3}$
(crit pts $x = y = \sqrt[3]{1}\, T^{A/3}$)

non-toric mirror 

$$\mathbb{C}^2\setminus\{uv=1\} \xrightarrow{} \mathbb{C} \quad u + \frac{T^A v^2}{uv-1}$$

each fiber has one more point!

The relation between the two geometries can be seen as coming from

$$(\mathbb{C}^*)^2 \underset{\text{dense}}{\overset{\text{open}}{\hookrightarrow}} \mathbb{C}^2 \setminus \{uv=1\}$$

$$(x,y) \longmapsto (u,v) = (x+y, x^{-1})$$

($\hookleftarrow (\mathbb{C}^*)^2_{(u,\, uv-1)}$ another embedding, giving another $(\mathbb{C}^*)^2$-chart, superpot Laur. poly)

the image is complement of a section (so each fiber in non-toric mirror has one more pt)

Could further compactify to

$(\mathbb{CP}^2, D = \text{smooth cubic})$, fibers $=$  elliptic curves

General principle: Fibers are mirror to $D$. |  mirror to $D = T^2$.

In this case,  mirror to $D =$ ;  mirror to partially smoothed $D =$ , There are a lot of charts $(\mathbb{C}^*)^2$, w/ potentials given by Laurent polynomials.

The mirror of $\mathbb{CP}^2$ rel smooth $D$ has a "cluster structure", i.e. a collection of distinguished $(\mathbb{C}^*)^2$ charts, related by birational transfs, st expression of $W$ is a Laurent poly in all these charts.

These charts $\leftrightarrow$ monot'd Lagr Tori in $\mathbb{CP}^2$ [Vianna]

$\leftrightarrow$ toric degenerations of $\mathbb{CP}^2$ to $\mathbb{CP}^2(a^2,b^2,c^2)$, indexed by $(a^2,b^2,c^2)$ for Markov triples $(a,b,c)$ : $a^2+b^2+c^2=3abc$ [Hacking-Prokhorov]

Conj: These are all monotone tori in $\mathbb{CP}^2$. Not clear for other mflds...

(Corti + Coates + Galkin...: get classification of Fano 3 folds by studying "reasonable" Laurent polynomials: w/ integer coeffs and that transform into other such Laurent polys under mutation)

---

HMS #1:

$\mathcal{F}(X)$ is $A_\infty$-deform (curved: $\mu^0$, not $\mathbb{Z}$-graded) of $\mathcal{F}(X^0)$, described by $\alpha_D \in HH^0(\mathcal{F}(X^0)) \leftrightarrow \mathcal{O}(X^\vee) \ni W$.

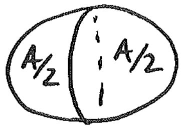
$\mathcal{F}(X)_\lambda$ (uncurved) $A_\infty$-categ,

Objs = weakly unobstructed Lagrs in $\mathcal{F}(X)$, st $\mu^0 = \lambda\, id$.

HMS:

$$D^\pi \mathcal{F}(X)_\lambda \simeq \underset{\substack{\| \\ \mathrm{Coh}/\mathrm{Perf}}}{D^\pi_{sing}(W^{-1}(\lambda))} \underset{\text{Orlov}}{\cong} MF(W-\lambda) \quad (=0 \text{ if } \lambda \notin \mathrm{Critv}(W))$$

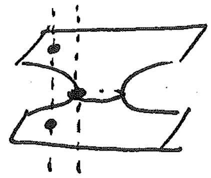
**Ex:** $\mathbb{CP}^1$




$\longleftrightarrow \quad W - \lambda = z + \dfrac{T^A}{z} \mp 2T^{A/2}$

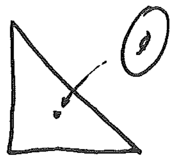
hol $= \pm 1$

$\lambda = \pm 2 T^{A/2}$



$$= (z \mp T^{A/2})(1 \mp z^{-1} T^{A/2})$$

**Ex:** $\mathbb{CP}^2 \supset T_{\text{Clifford}} = \{ (x:y:z) \mid |x| = |y| = |z| \}$




w/ 3 local systems, st $HF(T,T) \simeq H^*(T^2)$ as vect space

(ring: Clifford algebra $= \mathrm{Mat}_{2\times 2}$)

(other fibers are displaceable)

$\lambda = 3 \sqrt[3]{1}\, T^{A/3}$

The 3 crit pts of $W = x + y + \dfrac{T^A}{xy}$ give rise to MF's.

**Exercise:** $D^b_{sing}(\{xy=0\} \subset \mathbb{C}^2)$ vs $MF(\mathbb{C}^2, xy)$

$\quad\quad \text{"} = \mathrm{Coh}/\mathrm{Perf}$

$\quad\quad\quad \llcorner$ sheaves given by finite complexes of f.d. vector bdles

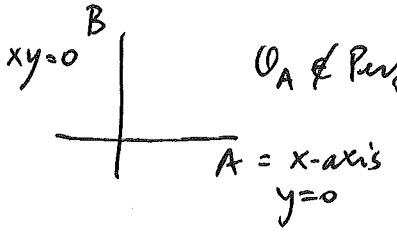
No finite resolution of $\mathcal{O}_A$ by vector bdles: for example

$\mathcal{O} = R = \mathbb{C}[x,y]/xy$

$\mathcal{O}_A = R/y = \mathbb{C}[x]$

$$\underbrace{\cdots \to \mathcal{O} \xrightarrow{y} \mathcal{O} \xrightarrow{x} \mathcal{O} \xrightarrow{y} \mathcal{O}}_{\text{resolution eventually 2-periodic}} \to \mathcal{O}_A \to 0$$

$\searrow$ MF $\left[ \mathcal{O} \underset{x}{\overset{y}{\rightleftarrows}} \mathcal{O} \right]$




$\mathcal{O}_A \notin \mathrm{Perf}$

Have SES

$$0 \to \mathcal{O}_B \to \mathcal{O} \to \mathcal{O}_A \to 0$$

$$0 \to \mathcal{O}_A \oplus \mathcal{O}_B \to \mathcal{O} \to \mathcal{O}_{p=0} \to 0$$

$\Rightarrow$ in $D^b_{sing}$, $\mathcal{O}_B \simeq \mathcal{O}_A[1]$, $\mathcal{O}_{p|=0} \simeq \mathcal{O}_A \oplus \mathcal{O}_B[1]$

$$\mathrm{End}(\mathcal{O}_{p|=0}) \simeq \mathrm{Mat}_{2\times 2} \simeq H^*(T^2) \text{ deformed}$$

—

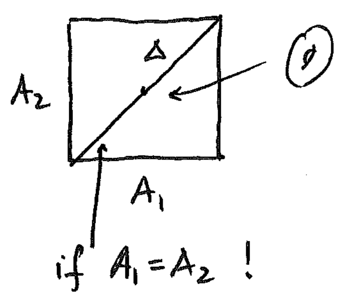
<u>Ex</u>: toric Fano: AFOOO

$\mathcal{F}(X)_\lambda$ split-generated by Lagr tori (w/ local systs & bounding cochains)

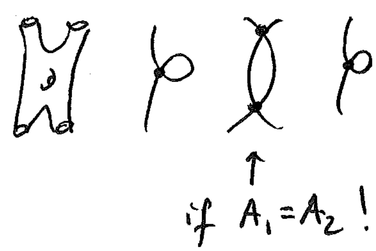
For generic Kähler class $[\omega]$, $\mathcal{F}(X)$ consists of tori
$\Longleftrightarrow$ mirror $W$ has isolated non-deg crit pts (as before)

For specific $[\omega]$, situation might be richer:

$\mathbb{CP}^1 \times \mathbb{CP}^1$
$\cup$
$(T = S^1_{eq} \times S^1_{eq})_{\pm\pm}$

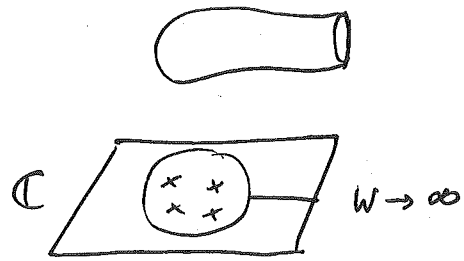



$$W = x + y + \frac{T^{A_1}}{x} + \frac{T^{A_2}}{y}$$



# Other side of HMS : $D^b \mathrm{Coh}(X) \simeq FS(X^\vee, W)$

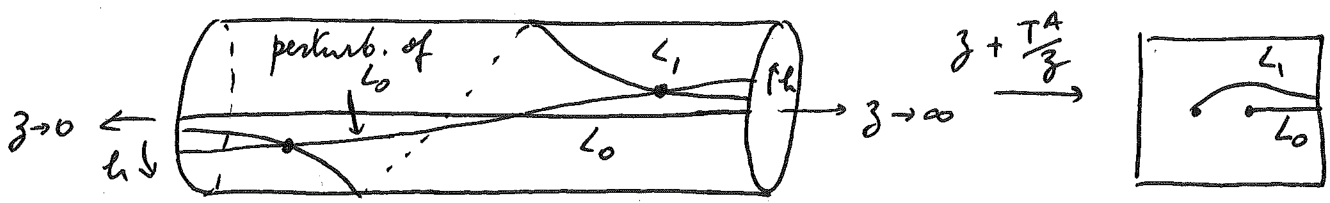
FS: $L \subset X^\vee$
going to right-ish




homs: Lagr Floer theory w/ Hamilt h st in base 


(if $W|_L$ not proper, wrap in fiber directions
- so, wrap in fibers and rotate a little in base)

Ex: $(\mathbb{C}^*, z + \frac{T^a}{z})$




$\mathrm{End}(L_i) = \mathbb{K}$

$\mathrm{Hom}(L_1, L_0) = 0$, $\mathrm{Hom}(L_0, L_1) \simeq \mathbb{K}^2$ $\longleftrightarrow$ $D^b \mathrm{Coh}(\mathbb{CP}^1)$

$$\mathcal{O} \overset{\mathbb{K}^2}{\rightrightarrows} \mathcal{O}(1)$$

id id

Seidel: the two thimbles generate FS

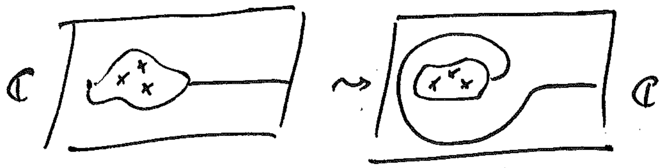
Beilinson exceptional collection

HMS can be proved in this way:

- Del Pezzo surfaces [A-Katzarkov-Orlov]
- Toric varieties [Abouzaid; FLTZ] in slow

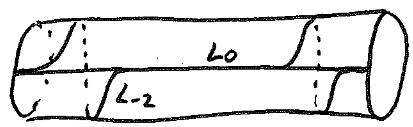
## Have several functors

1) <u>One-turn wrapping</u> : $FS(X^\vee, W) \circlearrowleft \phi_1$




symplecto supported in region over annulus

in $\mathbb{C}^*$ :




This gives an autoequivalence of caty, since can undo it, Twisting the other way.

This comes w/ a natural transformation $\phi_1 \xrightarrow{\sigma} id$

$\updownarrow$ mirror to

(use continuation map assoc to going from $H=0$ to $H$ linear at $\infty$ implementing $\phi_1$ - get disting. elt in $hom(\phi_1(L), L)$ $\forall L$

[Seidel]

$D^b Coh(X) \circlearrowleft - \otimes \mathcal{O}(-D)$
$(= K_X)$

+ natural transf. $- \otimes \mathcal{O}(-D) \xrightarrow[\times s_D]{} id$ (multiply by defining section for $D$)

2) <u>Acceleration functor</u> : $FS(X^\vee, W) \to \mathcal{W}(X^\vee)$

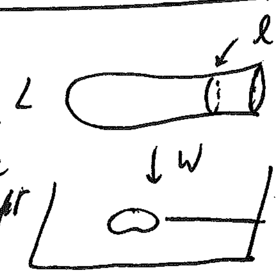
($\nearrow$ Ham, or take $\varinjlim$ over linear Hams)

ie <u>localization</u> wrt $\phi_1 \xrightarrow{\sigma} id$ (if do $\phi_1$ more and more, get $\mathcal{W}$)

$\updownarrow$ mirror

restriction $D^b Coh(X) \to D^b Coh(X \setminus D)$ (localize wrt $\times s_D$).

3) <u>Restriction to fiber</u>




fibers over arc in base, outside a cpt

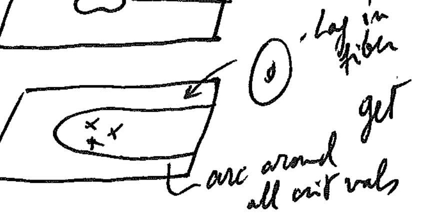
get $FS(X^\vee, W) \to \mathcal{F}(fiber)$ / if fiber not cpct
(or $\mathcal{W}(fiber)$ - not yet defined...)

mirror $\updownarrow$

$D^b Coh(X) \underset{i_*}{\overset{i^*}{\rightleftarrows}} D^b Coh(D)$ (derived)

$i^*, i_*$ are adjoint functors

Orlov: mirror to $i_*$




get $\mathcal{F}(fiber) \to FS(X^\vee, W)$