So,

$$\boxed{\hom_{FS}(L_i, L_j) \longleftrightarrow \mathrm{Ext}_{Coh(\mathbb{P}^1)}(\mathcal{O}(i), \mathcal{O}(j))}$$

Observe: $\phi_h^1(L_i) \simeq L_{i+2}$

$$CW^*_{lin}(L_i, L_j) = \varinjlim_{w\to\infty} CF^*(\phi_{wh}(L_i), L_j) = \varinjlim CF^*(L_{i-2w}, L_j) \quad (**)$$

What is this mirror to in Alg. Geom.?

$$\mathbb{C}^* = \mathbb{CP}^1 \setminus \overbrace{\{0,\infty\}}^{D}$$

$$\hom_{\mathbb{C}^*}(\mathcal{O}(i)|_{\mathbb{C}^*}, \mathcal{O}(j)|_{\mathbb{C}^*}) = \varinjlim_{w\to\infty} \hom_{\mathbb{CP}^1}(\mathcal{O}(i-2w), \mathcal{O}(j))$$

connecting map: multipl. by defining eqn of $D = \{0,\infty\}$

This matches the connecting map in $(**)$ above!

So, $\mathbb{C}^* \subset \mathbb{CP}^1$

$$\big\updownarrow \text{HMS}$$

$\mathbb{C}^*, W \leftrightarrow (\mathbb{C}^*, z + \frac{1}{z})$, FS

Can also do partial wrappings (Sylvan).

Eg: if if at 0 and wrap at $\infty$ in $\mathbb{C}^*$, recover $W(\mathbb{C})$.

10/27

## (Partialy) compactify a CY manifold

<u>Ex</u> : $\mathbb{C}^* = \mathbb{CP}^1 \setminus \{0, \infty\}$

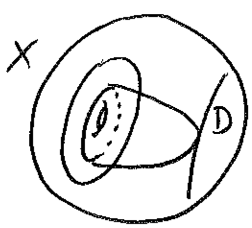
$X$ Kähler, $D \subset X$ w/ normal crossings singularities, reduced (no multiplicities), $|D| = -K_X$ ($\Rightarrow [D] = c_1(X)$).

$X^\circ := X \setminus D$ is (open) CY ($K_{X^\circ} = 0$).

$\mathcal{F}(X)$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $A_\infty$-deformation of $\mathcal{F}(X^\circ)$.

Expect to have enough Lagrs in $X^\circ$ to capture $\mathcal{F}(X)$ (morally, recall can make Lagrs disjoint from high degree $D$).

Let $L \subset X^\circ$ be a Lagrangian that is unobstructed in $X^\circ$.
Count disks in $(X, L)$ w/ $[u] . [D] = k$ w/ coeff $q^k$.




## "Relative" Fukaya category of $(X, D)$ over $\mathbb{K}[[q]]$ (Seidel, Sheridan)

can be $\Lambda$

$\rightsquigarrow$ set $q = 0$ : recover $\mathcal{F}(X^\circ)$

set $q = T^R$ : recover $\mathcal{F}(X, [\omega] + R[D])$

First order deformation ($q^1$ terms) gives deformation class
$\alpha_D \in HH^*(\mathcal{F}(X^0))$ (graded, filtered algebra)

( $CC^*(\mathcal{F}) = \prod \hom\left( CF(L_{k-1},L_k)^{[1]} \otimes \dots \otimes CF(L_0,L_1)^{[1]}, CF(L_0,L_k)\right)$
w/ differential & product )

If deformation preserves $\mathbb{Z}$-grading, then $\alpha_D \in HH^2(\mathcal{F}(X^0))$
$\mu^k_q = \mu^k_0 + q\mu^k_1 + \dots$
$\quad\quad\quad$ ↖ deg $2-k$ operation

Assuming $L$ had vanishing Maslov in $X^0$ (ie $L$ $\mathbb{Z}$-graded in $X^0$),
Maslov index of disk in $X$ w/ bdry in $L$ = $2 \cdot$ int # w/ $D$.

$\mathbb{Z}$-grading ok if $\deg(q) = 2$.
I.e., $\mu^k_1$ has deg $-k$ instead of $2-k$
$\rightsquigarrow \alpha_D \in HH^0(\mathcal{F}(X^0)) \simeq HH^0(D^bCoh(X^\vee)) \overset{HKR}{=} H^0(X^\vee, \Lambda^0 T_{X^\vee}) = \mathcal{O}(X^\vee)$
$\quad\quad$ ↑ mirror to $X^0$ (say a quasiproj. variety)

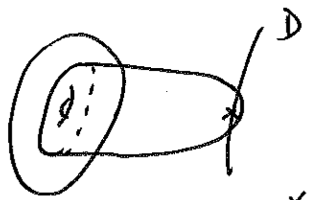
Concretely, equip $X^\vee$ w/ a function $W \in \mathcal{O}(X^\vee)$ "<u>superpotential</u>".
If $X^\vee$ is mirror to $X^0$, expect $(X^\vee, W)$ to be mirror to $X$.

Abouzaid: higher order defs correspond to higher index discs, which don't give rigid operations. They contribute to negative degree $HH^*$. Sheridan: related to orbifold compactifs.

# (Higher index disks might intersect divisor at more pts, or w/ multiplicities.)

<u>Note</u>: As we'll see, the superpotential $W \in \mathcal{O}(X^\vee)$ is st its MF category should be equivalent to $\mathcal{F}$ w/ weakly <u>unobstructed</u> Lags
$\quad\quad$ ↳ related to $W$

Skyscraper sheaves of pts in $X^\vee$ correspond to Lagrs in $X^0$ (typically tori).

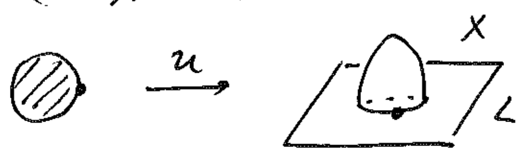



$(L, \xi) \longleftrightarrow p \in X^\vee$

$$\boxed{W(p) = \sum_{\substack{\beta \in \pi_2(X,L) \\ \beta\cdot D = 1}} n_\beta \, T^{\omega(\beta)} \mathrm{hol}(\partial\beta) \in \mathbb{K}}$$

this is an analytic function on the moduli space of Lagr tori $X^\vee$
"Torus-like objects in Fuk"

$$n_\beta = \deg\left(\mathcal{M}_{0,1}(L,\beta) \xrightarrow{ev} L\right) \in \mathbb{Z}$$




expected dim
$n - 3 + \underbrace{\mu(\beta)}_{2} + 1 = n$

Ex:

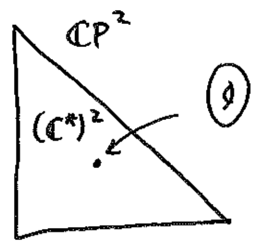

$(L, \nabla)$

$\mathbb{CP}^1 \longleftrightarrow \mathbb{K}^*$, $W = T^a \mathrm{hol} + T^{A-a}\mathrm{hol}^{-1}$
$= z + \dfrac{T^A}{z}$, $z = T^a \mathrm{hol}$

Note: don't get all of $\mathbb{K}^*$, but just a bounded domain therein (on which do inflation ...)

Ex: $X$ toric Fano variety (or semi-positive)
$= (\mathbb{C}^*)^n \cup D$
$D$ union of toric strata

Eg:

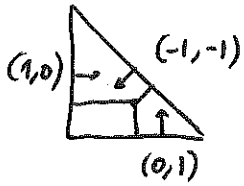



$\big(S^1(r_1) \times S^1(r_2), \nabla\big) \subset \mathbb{C}^2$, $\longleftrightarrow$ pts of $(\mathbb{K}^*)^2$

disks: $D^2(r_1) \times \{pt\} \rightsquigarrow z_1 = T^{a_1}\mathrm{hol}_1$
$\{pt\} \times D^2(r_2) \rightsquigarrow z_1 = T^{a_2}\mathrm{hol}_2$

$(z_1, z_2)$ coords in $(\mathbb{K}^*)^2$.

Complex analysis (maximum principle) $\Rightarrow$ one disc for each component in $D$




$$z_1 = T^{a_1}\,hol_1$$
$$z_2 = T^{a_2}\,hol_2$$
$$z_3 = \frac{T^{Area(\mathbb{CP}^1)}}{z_1 z_2}$$

$$W = z_1 + z_2 + \frac{T^A}{z_1 z_2}$$

More generally, for $X$ toric Fano,

$W$ = Laurent polynomial, where monomials $\leftrightarrow$ facets of moment polytope

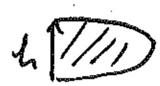
(In semi-positive case, can also have contributions from Maslov 2 discs # Chern 0 sphere.
Get power series, rather than polys, involving GW invts.
More general case is even more complicated.)

Obstruction: when $L \subset X$ bounds holom discs, $\boxed{\mu_L^0 \in CF^{*=2}(L,L)}$

$\mathcal{M}_{0,1}(L,\beta) = \{$  $\xrightarrow{u} (X,L) \mid \bar\partial_J u = 0,\ [u]=\beta \} / Aut(D^2, \bullet)$

$$\sum_\beta T^{\omega(\beta)}\, hol\ ev_*[\mathcal{M}_{0,1}(L,\beta)] \in C_{n-*}(L,\Lambda)$$

as a current on $L$ (as a diff. form? as a coho class?)

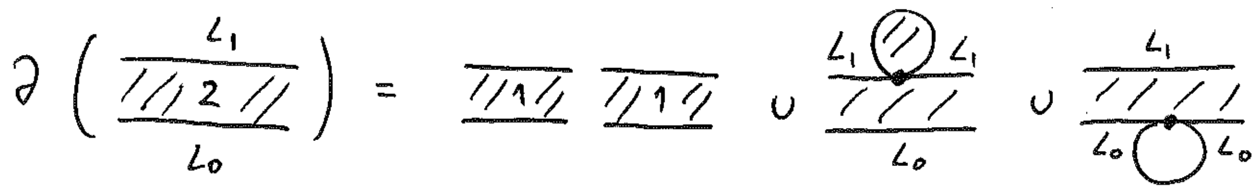
- or: $h$  $\in CF^*(L,L;h)$
- or: "pearly"  $\in CM^*(h) \simeq CF^*(L,L)$

Disk bubbling is accounted for by adding $\mu^0$ in $A_\infty$-eq:

$$\sum_{\ell \geq 0} (-1)^{\cdots} \mu^{k-\ell+1}(\dots, \mu^\ell(\dots), \dots) = 0.$$

Now, given $x \in CF^*(L_0, L_1)$ (and "choosing" signs)

$$\mu^1(\mu^1(x)) = \mu^2(\mu^0_{L_1}, x) - \mu^2(x, \mu^0_{L_0}) :$$




Hard approach: use stabilizing divisor, as in Cieliebak-Mohnke.

There are many regularity issues associated to disk-bubbling

<u>Def</u>: $L$ is <u>weakly unobstructed</u> if $\mu^0_L = \underset{\in}{\overset{\wedge}{w}} \cdot id$.

<u>Eg</u>: • no holom disks of Maslov $\leq 0$ (or cancel algebraically)

and • $\mu = 2$ are regular, give an $n$-chain, automatically a cycle $w.[L]$.

If weakly unobstructed, then $\boxed{(\mu^1)^2 = (w_1 - w_2)\, id}$.

Back to case of $L \subset X^0 = X \setminus D$ above,

$L$ is weakly unobstructed, with $\mu^0 = W \cdot id$

$\mathcal{F}_w(\mathbb{CP}^1) = \{$ weakly unobstructed Lagrs st $\mu^0 = w \cdot 1\}$

is an honest $A_\infty$-categ. for all $w \in \mathbb{K}$.


$(L, \nabla)$

$$\mu^0 = W(z) \cdot \mathrm{id}$$
$$W(z) = z + \frac{T^A}{z}$$

Have an object of $\mathcal{F}_{W(z)}$, but get $HF(L,L) = 0$ unless $L$ is an equator: $a = \frac{A}{2}$. (otherwise, $L$ is displaceable!)

Also must have hol $\pm 1$.

So, non-trivial objs ($HF(L,L) \neq 0$) are at $z = \pm T^{A/2}$.

$\rightsquigarrow W = z + \frac{T^A}{z} = \pm 2 T^{A/2}$.

These are the <u>critical pts</u> of $W$.

<u>Mirror:</u> $MF_w(X^\vee, W)$ matrix factorizations.

Assume: $X^\vee = \operatorname{Spec} R$ affine.

$$R^{\oplus k} \underset{\partial}{\overset{\partial}{\rightleftarrows}} R^{\oplus k}, \quad \boxed{\partial^2 = (W - w) \cdot \mathrm{id}} \quad \underline{\text{matrix factorization}}$$

$MF(\mathbb{K}^*, z + \frac{T^A}{z} - w)$ non-trivial iff $w = \pm 2 T^{A/2}$.

$$\mathbb{K}[z^{\pm 1}] \underset{g}{\overset{f}{\rightleftarrows}} \mathbb{K}[z^{\pm 1}] \qquad f \cdot g = W - w$$

$$z + \frac{T^A}{z} \pm 2 T^{A/2} = (z - T^{A/2})\left(1 - \frac{T^{A/2}}{z}\right)$$

This non-trivial matrix fact is mirror to equator w/ hol $= \pm 1$.

<u>Orlov:</u> $MF(W-w) \simeq D^b\mathrm{Sing}(W-w) := D^b\mathrm{Coh}(W^{-1}(w)) / \mathrm{Perf}$